

# COBORDISM CATEGORIES AND MODULI SPACES OF ODD DIMENSIONAL MANIFOLDS

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**ABSTRACT.** We prove that stable moduli spaces of highly-connected,  $(2n+1)$ -dimensional manifolds are homology equivalent to infinite loopspaces for  $n \geq 4, n \neq 7$ . The main novel ingredient is a version of the cobordism category incorporating surgery data in the form of lagrangian subspaces. Our main result can be viewed as a first step toward obtaining an analogue of the Madsen-Weiss theorem for odd dimensional manifolds.

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## 1. INTRODUCTION

**1.1. History and Motivation.** The study of diffeomorphisms of smooth manifolds has been a focus of differential topology from its inception. After initial geometric techniques were developed, the method of choice for attacking such automorphism groups was a combination of surgery theory and Waldhausen's A-theory, with the former providing input on block diffeomorphisms and the latter on the difference between block and honest diffeomorphisms. With Tillmann's work from [30] on cobordism categories of surfaces and Madsen and Weiss' proof of the Mumford conjecture in [18], however, a new method emerged, whose application to high dimensional manifolds was pioneered by Galatius and Randal-Williams in [8] and [9]. Their work focuses on manifolds of even dimension and

in the most basic case proceeds roughly as follows. A refinement of Quillen's homological stability method [25] is applied to the sequence

$$\mathrm{BDiff}(S^n \times S^n, D^{2n}) \longrightarrow \mathrm{BDiff}((S^n \times S^n)^{\#2}, D^{2n}) \longrightarrow \mathrm{BDiff}((S^n \times S^n)^{\#3}, D^{2n}) \longrightarrow \dots$$

to show that its homology stabilizes in degrees linearly bounded by the number of additional summands, see [11]. In a completely independent second step they show that the *scanning map*

$$(1.1) \quad \operatorname{colim}_{g \rightarrow \infty} \mathrm{BDiff}((S^n \times S^n)^{\#g}, D^{2n}) \longrightarrow \Omega_0^\infty \mathrm{MT}\theta^n$$

induces a homology isomorphism. The right-hand side is the infinite loop space of a certain Thom spectrum and thus its (co)homology is readily computable by the standard techniques of algebraic topology.

A similarly complete description of the homology of diffeomorphism groups in the case of odd dimensional manifolds is as of yet conjectural at best. Homological stability for the diffeomorphism groups of certain odd dimensional manifolds was recently established by the second author in [22] and [23]. The present paper should be regarded as a first step towards computing the stable homology of the diffeomorphism groups of odd dimensional manifolds by similar methods as in [9]. To explain the status quo and motivate our results, we begin by presenting a more detailed sketch of the techniques from [9], [8] and [7] used to establish the homology equivalence (1.1).

**1.2. The even dimensional picture.** Fix a map  $\theta : B \longrightarrow BO(d)$ . A crucial role is played by the cobordism category  $\mathbf{Cob}_\theta$  defined in [7], which we first recall. Objects of  $\mathbf{Cob}_\theta$  are given by  $(d-1)$ -dimensional, closed submanifolds  $M \subset \mathbb{R}^\infty$ , equipped with a bundle map  $\ell_M : TM \oplus \epsilon^1 \longrightarrow \theta^* \gamma^d$ , i.e. a  $\theta$ -structure. A morphism between objects  $M$  and  $N$  is given by a  $d$ -dimensional embedded cobordism  $W \subset [0, t] \times \mathbb{R}^\infty$ , equipped with a bundle map  $\ell_W : TW \longrightarrow \theta^* \gamma^d$  that restricts to  $\ell_M$  and  $\ell_N$  over the boundary. The category  $\mathbf{Cob}_\theta$  is topologized so that for each  $M, N \in \mathrm{Ob} \mathbf{Cob}_\theta$  there is a weak homotopy equivalence

$$\mathbf{Cob}_\theta(M, N) \simeq \coprod_W \mathrm{BDiff}_\theta(W, M \sqcup N),$$

where the union ranges over all diffeomorphism classes of compact manifolds  $W$  equipped with a specified identification  $\partial W \cong M \sqcup N$ . The main theorem from [7] yields a weak equivalence

$$(1.2) \quad \Omega B \mathbf{Cob}_\theta \xrightarrow{\simeq} \Omega^\infty \mathrm{MT}\theta,$$

where  $\mathrm{MT}\theta$  is the Thom spectrum associated to the virtual vector bundle  $-\theta^* \gamma^d$  over  $B$ .

With the cobordism category in place we can now describe how to obtain the homology equivalence from (1.1). To this end consider the space

$$\mathcal{M}_{2n} := \coprod_W \mathrm{BDiff}(W, D^{2n}),$$

with union ranging over the  $(n-1)$ -connected,  $n$ -parallelizable, closed manifolds, of dimension  $2n$ . Suitably modeled, the space  $\mathcal{M}_{2n}$  has the structure of a homotopy commutative topological monoid,

the operation induced by connected sum. Let  $\theta^n : BO(2n)\langle n \rangle \rightarrow BO(2n)$  denote the  $n$ -connected cover and consider the category  $\mathbf{Cob}_{\theta^n}$ . The space of  $\theta^n$ -structures (with fixed boundary behavior) on any  $(n-1)$ -connected,  $n$ -parallelizable manifold is weakly contractible. On the one hand, it is shown in [9] that the group completion theorem of McDuff and Segal from [20] can be applied to derive a homology equivalence

$$(1.3) \quad \operatorname{colim}_{g \rightarrow \infty} \operatorname{BDiff}((S^n \times S^n)^{\#g}, D^{2n}) \longrightarrow \Omega_0 B\mathcal{M}_{2n}.$$

On the other hand, by removing two disks from the given fixed disk, one obtains a multiplicative map

$$(1.4) \quad \mathcal{M}_{2n} \longrightarrow \mathbf{Cob}_{\theta^n}(S^{2n-1}, S^{2n-1}),$$

where the right-hand side is the endomorphism monoid on the standard sphere  $S^{2n-1} \subset \mathbb{R}^\infty$ , equipped with its essentially unique  $\theta^n$ -structure  $\ell_{S^{2n}}$ .

The scanning map of (1.1) can then be factored as follows:

$$(1.5) \quad \begin{array}{ccc} \Omega B\mathcal{M}_{2n} & \xrightarrow{\quad} & \Omega^\infty \operatorname{MT}\theta^n \\ \downarrow & & \uparrow \simeq \\ \Omega B\mathbf{Cob}_{\theta^n}(S^{2n-1}, S^{2n-1}) & \longrightarrow & \Omega B\mathbf{Cob}_{\theta^n} \end{array}$$

where the lower morphism is induced by taking an endomorphism to its associated loop in the first skeleton of the classifying space. The majority of the technical work in [9] is devoted to establishing the weak homotopy equivalence

$$(1.6) \quad \Omega B\mathcal{M}_{2n} \xrightarrow{\simeq} \Omega B\mathbf{Cob}_{\theta^n},$$

where the map is the one given by the bottom-horizontal arrow, which is induced by (1.4). This is achieved via a sequence of parametrized surgery arguments, making first the morphisms and then the objects ever higher connected. Putting these equivalences (1.2), (1.3), and (1.6) together, it follows that the top-horizontal map of (1.5) is a homology equivalence (in the limit as  $g \rightarrow \infty$ ), and thus yields the main theorem of [9], and the *Madsen-Weiss Theorem* [18] in the  $n = 1$  case.

**1.3. The odd-dimensional case.** It is tempting to try to carry out a similar program to study the stable moduli spaces of odd-dimensional manifolds. As before we define a homotopy commutative topological monoid

$$(1.7) \quad \mathcal{M}_{2n+1} := \coprod_W \operatorname{BDiff}(W, D^{2n+1}),$$

with union ranging over diffeomorphism classes of  $(n-1)$ -connected,  $n$ -parallelizable,  $(2n+1)$ -dimensional closed manifolds. For  $\theta^n : BO(2n+1)\langle n \rangle \rightarrow BO(2n+1)$ , one can still form an embedding of topological monoids  $\mathcal{M}_{2n+1} \hookrightarrow \mathbf{Cob}_{\theta^n}(S^{2n}, S^{2n})$  and apply the group completion theorem of [20]

to identify the homology of  $\Omega_0 B\mathcal{M}_{2n+1}$  with that of a suitably stabilized diffeomorphism group defined below. However, the arguments from [9] cannot be used to prove that the analogous map

$$\Omega B\mathcal{M}_{2n+1} \hookrightarrow \Omega B\mathbf{Cob}_\theta$$

is a weak homotopy equivalence, as the parametrized surgery techniques employed by Galatius and Randal-Williams fail when applied in the middle dimension of odd dimensional manifolds. This failure, however, is not a mere technicality as it had previously been established by Ebert in [6] that for an odd dimensional manifold the scanning map is never injective in rational cohomology, not even in a range. Thus, in order to complete the picture in odd dimensions, replacements for the cobordism category  $\mathbf{Cob}_\theta$  and the spectrum  $\mathbf{MT}\theta$  are needed.

A distinguishing and originally unexpected feature of the story in the  $2n$ -dimensional case is that the limiting space  $\operatorname{colim}_{g \rightarrow \infty} \operatorname{BDiff}((S^n \times S^n)^{\#g}, D^{2n})$  even has the homology type of an infinite loop space at all. Note that this is already implied by the equivalence  $\Omega B\mathcal{M}_{2n} \simeq \Omega B\mathbf{Cob}_{\theta^n}$ , since the cobordism category admits disjoint unions (and hence a symmetric monoidal structure) making  $B\mathbf{Cob}_{\theta^n}$  an infinite loop space. In the case  $n = 1$ , establishing this fact was the original insight of Tillmann in [30], laying the groundwork for the Madsen-Weiss theorem in [18]. It is therefore a natural question whether the stabilized moduli spaces of odd dimensional manifolds have the homology type of infinite loop spaces as well, or equivalently whether  $\Omega B\mathcal{M}_{2n+1}$  is an infinite loop space. An affirmative answer, of course, leads to the problem of describing the homotopy type of the associated spectrum in a computationally useful way.

**1.4. Statement of results.** We set out to answer the questions just raised by defining a substitute for  $\mathbf{Cob}_\theta$ . Let  $n \geq 4, n \neq 7$  and let  $\theta : B \rightarrow BO(2n+1)$  have  $n$ -connected source (these restrictions imply the existence of the quadratic form  $\mu$  used below). The following category is the main object of study:

**Definition.** *The topological category  $\mathbf{Cob}_\theta^\mathcal{L}$  has as its objects pairs  $(M, L)$  that satisfy the following conditions:*

- (i)  *$M$  is an object of  $\mathbf{Cob}_\theta$ , i.e.  $M \subset \mathbb{R}^\infty$  is a  $2n$ -dimensional closed submanifold equipped with a  $\theta$ -structure.*
- (ii)  *$L \leq H_n(M)$  is a Lagrangian subspace with respect to the intersection and selfintersection form  $(H_n(M), \lambda, \mu)$ . By Lagrangian we mean that  $L^\perp = L$  with respect to  $\lambda$  and  $\mu|_L = 0$ .*

*The morphism space  $\mathbf{Cob}_\theta^\mathcal{L}((M, L_M), (N, L_N))$  is the following subspace of  $\mathbf{Cob}_\theta(M, N)$ . A cobordism  $W \subseteq [0, t] \times \mathbb{R}^\infty$  from  $M$  to  $N$  is a morphism in  $\mathbf{Cob}_\theta^\mathcal{L}((M, L_M), (N, L_N))$  if:*

- (a) *The pair  $(W, N)$  is  $(n-1)$ -connected;*
- (b)  *$\iota^{in}(L_M) = \iota^{out}(L_N)$ , where  $\iota^{in} : H_n(M) \rightarrow H_n(W)$  and  $\iota^{out} : H_n(N) \rightarrow H_n(W)$  are the maps induced by the boundary inclusions.*

To state our main theorem, we restrict to the tangential structure given by the  $n$ -connected (!) cover  $\theta^n : BO(2n+1)\langle n \rangle \rightarrow BO(2n+1)$ . Recall the topological monoid  $\mathcal{M}_{2n+1}$  from (1.7). Since  $H_n(S^{2n}) = 0$ , all Lagrangian subspaces of  $H_n(S^{2n})$  are zero, and thus the forgetful map  $\mathbf{Cob}_n^{\mathcal{L}}(S^{2n}, S^{2n}) \cong \mathbf{Cob}_{\theta^n}(S^{2n}, S^{2n})$  is a homeomorphism. As with (1.4), we obtain a multiplicative map  $\mathcal{M}_{2n+1} \hookrightarrow \mathbf{Cob}_n^{\mathcal{L}}(S^{2n}, S^{2n})$ , and thus a map  $\Omega B\mathcal{M}_{2n+1} \rightarrow \Omega B\mathbf{Cob}_{\theta^n}^{\mathcal{L}}$ . The following theorem is our main result:

**Theorem A.** *Let  $n \geq 4$  be an integer except 7. Then the map*

$$\Omega B\mathcal{M}_{2n+1} \rightarrow \Omega B\mathbf{Cob}_{\theta^n}^{\mathcal{L}}$$

*just described is a weak homotopy equivalence.*

For any tangential structure  $\theta$ , the operation of disjoint union almost makes  $\mathbf{Cob}_{\theta}^{\mathcal{L}}$  into a symmetric monoidal category. Indeed, for two objects  $(M, L_M), (N, L_N) \in \text{Ob } \mathbf{Cob}_{\theta}^{\mathcal{L}}$ , the sum of subspaces  $L_M + L_N \leq H_n(M \sqcup N) \cong H_n(M) \oplus H_n(N)$  is again a Lagrangian subspace of  $H_n(M \sqcup N)$ . While not a symmetric monoidal structure (due to the cobordisms remembering their lengths) we show that the above construction endows  $B\mathbf{Cob}_{\theta}^{\mathcal{L}}$  with the structure of a special  $\Gamma$ -space. Applying the results of Segal from [28] we therefore obtain:

**Corollary B.** *Let  $n \geq 4$  be an integer except 7. Then  $\Omega B\mathcal{M}_{2n+1}$  is an infinite loop space.*

By applying the group completion theorem one may deduce information about stable diffeomorphism groups. The definition of such objects is unfortunately slightly more complicated than in the even dimensional case: Let  $\mathcal{W}_{2n+1} = \pi_0(\mathcal{M}_{2n+1})$  denote the set of diffeomorphism classes of oriented,  $(n-1)$ -connected,  $(2n+1)$ -dimensional,  $n$ -parallelizable, closed manifolds. Fix a system of generators  $\{W_i\}_{i \in \mathbb{N}}$  for  $\mathcal{W}_{2n+1}$ . Let  $\mathbf{B}_{\infty}$  denote the colimit of the direct system

$$\text{BDiff}(W_1, D^{2n+1}) \rightarrow \text{BDiff}(W_1^{\#2} \# W_2, D^{2n+1}) \rightarrow \text{BDiff}(W_1^{\#3} \# W_2^{\#2} \# W_1, D^{2n+1}) \rightarrow \dots$$

where the  $k$ -th arrow in the above direct system is induced by multiplication by the element

$$W_1 \cdot W_2 \cdots W_{k-1} \cdot W_k \in \mathcal{M}_{2n+1}.$$

By homotopy commutativity of  $\mathcal{M}_{2n+1}$ , the homotopy type of the limiting space  $\mathbf{B}_{\infty}$  does not depend on the choice of generating set. We will refer to the space  $\mathbf{B}_{\infty}$  as the *stable moduli space of highly-connected manifolds of dimension  $2n+1$* . By application of the group completion theorem, we obtain a homology equivalence,  $\mathbf{B}_{\infty} \rightarrow \Omega_0 B\mathcal{M}_{2n+1}$ . This homology equivalence yields the following corollary.

**Corollary C.** *Let  $n \geq 4$  be an integer except 7. Then the stable moduli space  $\mathbf{B}_{\infty}$  has the homology type of an infinite loop space.*

For  $W \in \mathcal{W}_{2n+1}$ , let  $\mathcal{P}_W : \text{BDiff}(W, D^{2n+1}) \rightarrow \Omega_0^{\infty} \text{MT}\theta^n$  denote the *scanning map*, see [9] or [6] for a definition. Let  $F : \Omega_0 B\mathbf{Cob}_{\theta^n}^{\mathcal{L}} \rightarrow \Omega_0^{\infty} \text{MT}\theta^n$  denote the map induced by applying  $\Omega_0(\_)$

to the composite  $BCob_{\theta^n}^{\mathcal{L}} \longrightarrow BCob_{\theta^n} \xrightarrow{\simeq} \Omega^{\infty-1}MT\theta^n$ , where the first arrow is induced by the forgetful functor. Using the above corollary we can relate the homomorphisms that these two maps induce on cohomology with rational coefficients.

**Corollary D.** *Let  $n \geq 4$  be an integer except 7. Then the kernel of the homomorphism*

$$F^* : H^*(\Omega_0^\infty MT\theta^n; \mathbb{Q}) \longrightarrow H^*(\Omega_0 BCob_{\theta^n}^{\mathcal{L}}; \mathbb{Q})$$

*is equal to the common kernel of the collection of the maps*

$$\mathcal{P}_W^* : H^*(\Omega_0^\infty MT\theta^n; \mathbb{Q}) \longrightarrow H^*(B\text{Diff}(W, D^{2n+1}); \mathbb{Q})$$

*for all  $W \in \mathcal{W}_{2n+1}$ .*

**Remark.** *It follows from the main result of [6] that the common kernel of the maps*

$$\mathcal{P}_W^* : H^*(\Omega_0^\infty MT\theta^n; \mathbb{Q}) \longrightarrow H^*(B\text{Diff}(W, D^{2n+1}); \mathbb{Q})$$

*is non-zero and thus by Corollary D the map  $F$  cannot be a homology equivalence.*

Finally we note, that all of the above results, suitably interpreted, stay valid for an arbitrary  $\theta : B \rightarrow BO(2n+1)$  with  $B$   $n$ -connected, such that  $\theta$  is *weakly once stable* (see Definition 3.4). In particular, for such a  $\theta$  the space  $\Omega B\mathcal{M}_\theta$  always is an infinite loopspace, see Section 4.3.

**1.5. Outlook.** Since the work of Galatius and Randal-Williams by now covers even dimensional manifolds subject only to simple connectivity (they may even be closed) an obvious goal is to reach a similarly complete level of understanding in odd dimensions. In the even dimensional case these results rest on two pillars:

The first is the identification of the stable homology type of the diffeomorphism group with that of a cobordism category without connectivity assumptions [9]. This boils down to considering non-highly connected tangential structures  $\theta$ .

In the odd-dimensional setting it is currently unclear how to define a version of  $\mathbf{Cob}^{\mathcal{L}}$  for general  $\theta$ . The reason is the non-existence of a selfintersection form at the homology level. A possible solution to these problems is suggested by the work of Ranicki, i.e. to work directly with the chains on a manifold rather than its homology. At the chain level all relevant structure is present without connectivity assumptions. Carrying out such a generalization, however, presents serious technical difficulties, although recent work of Steimle on cobordism categories of Poincaré chain complexes indicates this may come within reach in the near future. We emphasize that such an approach would be complimentary to the techniques of this paper and would not replace them.

The second ingredient is the stable homological stability theorem of [12], which among other things allows one to replace the stabilization with a universal end by stabilization with just  $S^n \times S^n$ 's, for which homological stability is known; it just so happens that  $S^n \times S^n$ 's form a universal end for  $\theta^n$ , so this problem is not visible in the brief introduction above. An analogue of this result is work in progress and could replace stabilization with respect to all  $W_i \in \mathcal{W}_{2n+1}$ , by stabilization

with only  $S^n \times S^{n+1}$ , for which the second author proved homological stability in [22]. Such results improving on homological stability are largely independent of the present work.

Finally, the most important question to be addressed in future work is that of the homotopy type of our category  $\mathbf{Cob}_\theta^{\mathcal{L}}$  and the spectrum arising from it. One would expect such an analysis to be simpler than a direct analysis of the monoid  $\mathcal{M}_{2n+1}$  just as  $B\mathbf{Cob}_\theta$  can be identified with  $\Omega^{\infty-1}MT\theta$  using Pontryagin-Thom theory. While there are obvious guesses to be made, we cannot currently offer a concrete conjecture based on more than speculation.

The present article derives its relevance largely from bringing the determination of characteristic classes for odd dimensional manifold bundles within the scope of already developed machinery ([18], [7], and [9]) and showing that the remaining parts may not be too far off.

**1.6. Organization of the paper.** Section 2 is devoted to recollections of basic constructions from [9] and a new construction (Subsection 2.3) that enables us to topologize the category  $\mathbf{Cob}_\theta^{\mathcal{L}}$ . In Section 3 we define cobordism categories comprised of highly connected odd-dimensional manifolds. We construct the embedding  $\mathcal{M}_{2n+1} \hookrightarrow \mathbf{Cob}_{\theta^n}(S^{2n}, S^{2n})$  used in the statement of Theorem A and we also describe the relationship between  $\mathcal{M}_{2n+1}$  and the stable moduli space  $\mathbf{B}_\infty$ . In Section 4 we define the category  $\mathbf{Cob}_\theta^{\mathcal{L}}$  and show how to derive Theorem A and its corollaries from a number of technical results whose proofs make up the rest of the paper. After Section 5 constructs the  $\Gamma$ -space structure on  $B\mathbf{Cob}_\theta^{\mathcal{L}}$ , Sections 6 and 7 contain preliminary technicalities, which enable us to apply the parametrized surgery techniques of [9] in the final three sections.

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## 2. PRELIMINARIES ON SPACES OF MANIFOLDS

**2.1. Spaces of manifolds and cobordism categories.** We begin by reviewing some basic constructions from [8] and [9]. Recall that a *tangential structure* is a map  $\theta : B \rightarrow BO(d)$ . A



$\theta$ -structure on a  $d$ -dimensional manifold  $W$  is defined to be a bundle map  $TW \rightarrow \theta^*\gamma^d$  (i.e. a fibrewise linear isomorphism). More generally, a  $\theta$ -structure on an  $m$ -dimensional manifold  $M$  (with  $m \leq d$ ) is a bundle map  $TM \oplus \epsilon^{d-m} \rightarrow \theta^*\gamma^d$ . For the definition below, fix a tangential structure  $\theta : B \rightarrow BO(d)$ .

**Definition 2.1.** For an open subset  $U \subset \mathbb{R}^n$ , we denote by  $\Psi_\theta(U)$  the set of pairs  $(M, \ell)$  where  $M \subset U$  is a smooth  $l$ -dimensional submanifold (with empty boundary) that is closed as a topological subspace of  $U$ , and  $\ell$  is  $\theta$ -structure on  $M$ . More generally, if  $l \leq d$  is a nonnegative integer, we define  $\Psi_{\theta,l}(U)$  to be the set of pairs  $(M, \ell)$  where  $M \subset U$  is an  $l$ -dimensional submanifold without boundary and closed as a topological subspace, while  $\ell$  is a  $\theta$ -structure on  $M$ .

In [8, Section 2] the sets  $\Psi_{\theta,l}(U)$  were topologized so that the assignment  $U \mapsto \Psi_{\theta,l}(U)$  defines a pre-sheaf on  $\mathbb{R}^n$ , valued in topological spaces. We will recall and extend the construction of that topology below in Subsection 2.3. We will denote by  $\Psi_{\theta,l}(\mathbb{R}^\infty)$  the colimit of the spaces  $\Psi_\theta(\mathbb{R}^n)$  for  $n \rightarrow \infty$ . As in [8] we will need to consider particular subspaces of  $\Psi_{\theta,l}(\mathbb{R}^n)$  consisting of submanifolds  $M \subset \mathbb{R}^n$  that are open in a fixed number of directions.

**Definition 2.2.** For  $k \leq n$ ,  $\psi_\theta(n, k) \subset \Psi_\theta(\mathbb{R}^n)$  is the subspace consisting of those  $\theta$ -manifolds  $(M, \ell)$  such that  $M \subset \mathbb{R}^k \times (-1, 1)^{n-k}$ . The space  $\psi_\theta(\infty, k)$  is defined to be the direct limit of the  $\psi_\theta(n, k)$  taken as  $n \rightarrow \infty$ . For  $l \leq d$ , we make the analogous definition for  $\psi_{\theta,l}(n, k)$  and  $\psi_{\theta,l}(\infty, k)$ .

Using the spaces of manifolds defined above, we give a rigorous definition of the cobordism category  $\mathbf{Cob}_\theta$  discussed in the introduction. We will need to fix some notation. Let  $(M, \ell_M) \in \psi_{\theta,d-1}(\infty, 0)$ . The  $\theta$ -structure  $\ell$  on  $M$  determines a  $\theta$ -structure on the product  $\mathbb{R} \times M$  which we denote by  $\ell_{\mathbb{R} \times M}$ . In this way we have a continuous map

$$(2.1) \quad \mathbb{R} \times \_ : \psi_{\theta,d-1}(\infty, 0) \longrightarrow \psi_\theta(1 + \infty, 1), \quad (M, \ell) \mapsto (\mathbb{R} \times M, \ell_{\mathbb{R} \times M}).$$

All of our constructions will take place inside the product  $\mathbb{R} \times \mathbb{R}^\infty$ . We let

$$x_1 : \mathbb{R} \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$$

denote the projection onto the first factor. We will need to consider submanifolds  $W \subseteq \mathbb{R} \times \mathbb{R}^\infty$ . For any subset  $K \subseteq \mathbb{R}$ , we write

$$W|_K = W \cap x_1^{-1}(K)$$

If  $\ell$  is a  $\theta$ -structure on  $W$  and  $W|_K$  a submanifold of  $W$ , then we write  $\ell|_K$  for the restriction of  $\ell$  to  $W|_K$ .

**Definition 2.3.** We let the non-unital topological category  $\mathbf{Cob}_\theta$  have object space  $\psi_{\theta,d-1}(\infty, 0)$ . The morphism space is the following subspace of  $\mathbb{R} \times \psi_\theta(1 + \infty, 1)$ : A pair  $(t, (W, \ell))$  is a morphism if there exists an  $\varepsilon > 0$  with

$$W|_{(-\infty, \varepsilon)} = (-\infty, \varepsilon) \times W|_0 \quad \text{and} \quad W|_{(t-\varepsilon, \infty)} = (t-\varepsilon, \infty) \times W|_t$$



as  $\theta$ -manifolds, where  $(-\infty, \varepsilon) \times W|_0$  and  $(t - \varepsilon, \infty) \times W|_t$  are equipped with the  $\theta$ -structures induced from  $\ell|_0$  and  $\ell|_t$  on  $W|_0$  and  $W|_t$  using (2.1) and restricting. The source of such a morphism is  $W|_0$  and the target  $W|_t$ , equipped with their respective restrictions of the  $\theta$ -structure  $\ell$  on  $W$ .

**Remark 2.4.** From the results of [8], it follows that one has a weak homotopy equivalence

$$(2.2) \quad \text{Ob } \mathbf{Cob}_\theta \simeq \coprod_M \text{BDiff}_\theta(M),$$

with the disjoint union ranging over all closed,  $(d - 1)$ -dimensional manifolds  $M$ , and the space  $\text{BDiff}_\theta(M)$  explained below. For any two objects  $(M, \ell), (N, \ell') \in \text{Ob } \mathbf{Cob}_\theta$ , there similarly is a weak homotopy equivalence

$$(2.3) \quad \mathbf{Cob}_\theta((M, \ell), (N, \ell')) \simeq \coprod_W \text{BDiff}_\theta(W, M \sqcup N; \ell \sqcup \ell'),$$

where the union ranges over diffeomorphism classes (relative boundary) of compact manifolds  $W$ , equipped with a specified diffeomorphism  $\partial W \cong M \sqcup N$ . The space  $\text{BDiff}_\theta(W, M \sqcup N; \ell \sqcup \ell')$  is defined to be the homotopy quotient  $\text{Bun}(TW, \theta^* \gamma^d; \ell \sqcup \ell') // \text{Diff}(W, M \sqcup N)$ , where  $\text{Bun}(TW, \theta^* \gamma^d; \ell \sqcup \ell')$  is the space of all  $\theta$ -structures on  $W$  that agree with  $\ell \sqcup \ell'$  when restricted to  $\partial W$  under the specified diffeomorphism. The spaces  $\text{BDiff}_\theta(M)$  from (2.2) are defined similarly.

In [7], Galatius, Madsen, Tillmann, and Weiss determine the homotopy type of the classifying space of the cobordism category  $\mathbf{Cob}_\theta$ . Let us recall this result using the language of [8]. Let  $\text{MT}\theta$  denote the Thom spectrum associated to the  $-d$ -dimensional virtual vector bundle  $-\theta^* \gamma^d$  over  $B$ . Galatius and Randal-Williams construct a factorization of the Madsen-Tillmann scanning map

$$B\mathbf{Cob}_\theta \longrightarrow \Omega^{\infty-1} \text{MT}\theta$$

through the space  $\psi_\theta(\infty, 1)$  introduced above and show:

**Theorem 2.5.** *Both maps in the composition*

$$B\mathbf{Cob}_\theta \longrightarrow \psi_\theta(\infty, 1) \longrightarrow \Omega^{\infty-1} \text{MT}\theta$$

*are weak equivalences.*

**2.2. Homological preliminaries.** We will need to work with the homology groups of elements of  $\Psi_\theta(\mathbb{R}^\infty)$ , which are in general non-compact manifolds. This forces us to introduce some preliminary definitions, notation, and terminology.

**Definition 2.6.** Let  $X$  be a topological space. For each  $k \in \mathbb{Z}_{\geq 0}$ , the group  $H_k^{\text{cpt}}(X)$  is defined to be the inverse limit,

$$H_k^{\text{cpt}}(X) := \lim_{K \subseteq X} H_k(X, X \setminus K),$$

where the limit is taken over all compact subspaces  $K \subseteq X$ . The assignment  $X \mapsto H_k^{\text{cpt}}(X)$  is functorial with respect to proper maps between spaces.

For a pair of spaces  $(X, A)$  and  $k \in \mathbb{N}$ , the relative homology group  $H_k^{\text{cpt}}(X, A)$  is defined to be the inverse limit

$$H_k^{\text{cpt}}(X, A) := \lim_{K \subseteq X} H_k(X, A \cup (X \setminus K)),$$

with the limit taken over all compact subspaces  $K \subseteq X$ .

In the case of  $X$  a manifold, the description of these groups can be simplified as follows. Any compact subset of a manifold is contained in a compact codimension-0 submanifold (usually with boundary). Therefore, for any manifold  $X$  the group  $H_k^{\text{cpt}}(X)$  may be computed as the inverse limit of the groups  $H_k(X, X \setminus K)$ , taken over all compact codimension-0 submanifolds  $K \subset X$ . By excision,  $H_k(X, X \setminus K) \cong H_k(K, \partial K)$ , so we obtain

$$(2.4) \quad H_k^{\text{cpt}}(X) \cong \lim_{K \subseteq X} H_k(K, \partial K),$$

with the limit taken over all compact codimension-0 submanifolds  $K \subset X$ .

Let  $X$  be a manifold of dimension  $m$ . We will need to consider two types of restriction maps. First, let  $V \subset X$  be an open subset. The homomorphism

$$(2.5) \quad \pi_V : H_k^{\text{cpt}}(X) \longrightarrow H_k^{\text{cpt}}(V)$$

is defined to be the composition

$$H_k^{\text{cpt}}(X) \xrightarrow{\cong} \lim_{K \subseteq X} H_k(K, \partial K) \longrightarrow \lim_{K \subseteq V} H_k(K, \partial K) \xrightarrow{\cong} H_k^{\text{cpt}}(V),$$

where the first and last maps are the isomorphisms from (2.4), and the middle map is the natural projection onto the limit taken over the sub-inverse-system of all compact subsets contained in  $V$ . For the second type of restriction map, let  $j : Z \hookrightarrow X$  be the inclusion of a compact submanifold of dimension  $n \leq m = \dim(X)$ . Let  $U \subset X$  be a tubular neighborhood of  $Z$ . For all  $k \in \mathbb{Z}_{\geq 0}$  the homomorphism

$$(2.6) \quad j_! : H_k^{\text{cpt}}(X) \longrightarrow H_{k+n-m}(Z, \partial Z)$$

is defined to be the composition

$$H_k^{\text{cpt}}(X) \longrightarrow H_k(\bar{U}, \partial \bar{U}) \xrightarrow{\cong} H^{m-k}(\bar{U}) \xrightarrow{\cong} H^{m-k}(Z) \xrightarrow{\cong} H_{k+n-m}(Z, \partial Z),$$

where the first map is the canonical projection (of the inverse limit onto one of its factors), the second and fourth arrows are given by Lefschetz duality, and the middle map is given by restriction. Since any two tubular neighborhoods of  $Z$  are isotopic (as submanifolds of  $X$ ) it follows that the definition of the map  $j_!$  is independent of the choice of tubular neighborhood  $U$ . We will denote both of the above restriction maps (from (2.5) and (2.6)) by

$$x \longmapsto x|_Z.$$

**Remark 2.7.** This leads to very little ambiguity: If the open subset  $V \subseteq X$  is given as the interior of some compact codimension 0 submanifold  $Z \subseteq X$  we claim that  $H_k^{\text{cpt}}(V)$  and  $H_k(Z, \partial Z)$  are canonically isomorphic in a fashion making

$$\begin{array}{ccc} & H_k^{\text{cpt}}(X) & \\ \cdot|_V \swarrow & & \searrow \cdot|_Z \\ H_k^{\text{cpt}}(V) & \xleftarrow{\cong} & H_k(Z, \partial Z) \end{array}$$

commutative. To this end choose an open collar  $C$  of  $\partial Z$  in  $Z$ . As  $V - C$  is a compact subset of  $V$  we obtain a map  $H_k^{\text{cpt}}(V) \rightarrow H_k(V, C - \partial Z)$ , which we shall momentarily see is an isomorphism (since  $V$  has  $C - \partial Z$  as its cylindrical ends as defined below). Clearly,  $H_k(V, C - \partial Z)$  and  $H_k(Z, \partial Z)$  are canonically isomorphic (for example via their inclusions into  $H_k(Z, C)$ ). It is now readily checked that this identification is independent of the chosen collar and the diagram above indeed commutes.

Similar maps to (2.5) and (2.6) can be defined for the relative homology groups  $H_*^{\text{cpt}}(X, A)$  but we won't explicitly have to use them in the paper.

We will mainly work with a class of manifolds for which the groups  $H_*^{\text{cpt}}(X)$  simplify further.

**Definition 2.8.** A manifold  $X$  with  $\partial X = \emptyset$  is said to have *cylindrical ends* if there exists some compact codimension-0 submanifold  $B \subset X$  (possibly with boundary), such that the complement  $X \setminus \text{Int}(B)$  is diffeomorphic to the cylinder  $\partial B \times [0, \infty)$ , relative the boundary.

In the above definition the manifold  $X$  is homotopy equivalent to the compact manifold  $B$  and thus  $X$  is of finite type as a topological space. Such a codimension-0 compact submanifold  $B \subset X$  with this property will be referred to as a (codimension-0) *core* for  $X$ . It follows that the inclusion  $B \hookrightarrow X$  induces an isomorphism  $H_k^{\text{cpt}}(X) \cong H_k(B, \partial B)$ . This isomorphism, combined with Lefschetz duality yields the isomorphism

$$(2.7) \quad H_k^{\text{cpt}}(X) \xrightarrow{\cong} H_k(B, \partial B) \xrightarrow{\cong} H^{m-k}(B) \xrightarrow{\cong} H^{m-k}(X),$$

where  $m = \dim(X) = \dim(B)$ . We note that this identification does not depend on the choice of codimension-0 core  $B \subset X$ , though we will need not explicitly have to use this fact.

**2.3. Spaces of manifolds equipped with homological data.** We will need to consider spaces of manifolds equipped with a choice of subspace of its homology group. These spaces (defined below) will enable us to topologize the cobordism category  $\mathbf{Cob}_\theta^{\mathcal{L}}$  (discussed in the introduction) and the semi-simplicial spaces introduced in Section 6.

**Construction 2.1.** Fix a tangential structure  $\theta : B \rightarrow BO(d)$ . For an open subset  $U \subset \mathbb{R}^m$ , let  $\Psi_\theta^\Delta(U)$  denote the set of triples  $(M, \ell, V)$  with  $(M, \ell) \in \Psi_\theta(U)$  and  $V \leq H_*^{\text{cpt}}(M)$  a subgroup. For any non-negative integer  $l \leq d$ , we define  $\Psi_{\theta, l}^\Delta(U)$  similarly but with  $M$  an  $l$ -dimensional submanifold. We topologize the set  $\Psi_\theta^\Delta(U)$  by a three step process, analogous to the three step

process used in [8, Section 2.1] to topologize the spaces  $\Psi_\theta(U)$ .

**Step 1:** We begin by defining the *compactly supported topology*. We will write  $\Psi_\theta^\Delta(U)^{\text{cs}}$  for the set  $\Psi_\theta^\Delta(U)$  equipped with this topology. As with the definition of  $\Psi_\theta(U)^{\text{cs}}$  in [8, Page 6], a neighborhood of a point  $(M, \ell, V) \in \Psi_\theta^\Delta(U)^{\text{cs}}$  is homeomorphic to a neighborhood of the zero-section in the space  $\Gamma_c(\nu M)$  of compactly supported sections of the normal bundle  $\nu M$  of  $M \subset U$ . In order to do this we need to describe how to embed  $\Gamma_c(\nu M)$  into  $\Psi_\theta^\Delta(U)$  for each  $(M, \ell, V)$ .

Let  $(M, \ell, V) \in \Psi_\theta^\Delta(U)$ . We identify the normal bundle  $\nu M \rightarrow M$  with the sub-bundle of the trivial bundle  $\epsilon^m$  given by the orthogonal complement to the tangent bundle  $TM \subset \epsilon^m$  with respect to the metric induced from the standard Euclidean metric on  $\mathbb{R}^m$ . By the *tubular neighborhood theorem*, the exponential map  $\nu M \rightarrow U$  restricts to an embedding on some neighborhood of the zero section; let  $A \subset \nu M$  be such a neighborhood of the zero-section. Let  $\Gamma_c(\nu M, A) \subset \Gamma_c(\nu M)$  denote the subspace consisting of those sections with image contained in  $A \subset \nu M$ . Clearly  $\Gamma_c(\nu M, A) \subset \Gamma_c(\nu M)$  is an open subset. Given a section  $s \in \Gamma_c(\nu M, A)$ , the triple

$$(s(M), \ell \circ Ds, s_*(V)),$$

with  $s_*(V) \leq H_*^{\text{cpt}}(s(M))$ , and  $Ds : TM \rightarrow Ts(M)$  the differential of  $s$ , is an element of the set  $\Psi_\theta^\Delta(U)$ . This correspondence defines an injective map

$$c_{(M, \ell, V)} : \Gamma_c(\nu M, A) \rightarrow \Psi_\theta^\Delta(U)^{\text{cs}}, \quad s \mapsto (s(M), \ell \circ Ds, s_*(V)).$$

We topologize  $\Psi_\theta^\Delta(U)^{\text{cs}}$  by declaring the above maps to be homeomorphisms onto open sets. This makes  $\Psi_\theta^\Delta(U)^{\text{cs}}$  into an infinite dimensional manifold, modeled on the topological vector spaces  $\Gamma_c(\nu M)$ .

**Step 2:** For each compact subset  $K \subseteq U$ , let  $\Psi_\theta^\Delta(U \mid K)$  be the quotient space obtained by identifying two elements  $(M, \ell, V), (M', \ell', V') \in \Psi_\theta^\Delta(U)^{\text{cs}}$  if there exists some open neighborhood  $A \subset U$  of  $K$  for which

$$(M \cap A, \ell|_{M \cap A}, V|_{M \cap A}) = (M' \cap A, \ell'|_{M' \cap A}, V'|_{M' \cap A}),$$

where  $V|_{M \cap A} \in H_*^{\text{cpt}}(M \cap A)$  and  $V'|_{M' \cap A} \in H_*^{\text{cpt}}(M' \cap A)$  are the images of  $V$  and  $V'$  under the restriction maps from (2.5). Let

$$(2.8) \quad \pi_K : \Psi_\theta^\Delta(U)^{\text{cs}} \rightarrow \Psi_\theta^\Delta(U \mid K)$$

be the quotient map. We define  $\Psi_\theta^\Delta(\mathbb{R}^m)^K$  to be the topological space with the same underlying set as  $\Psi_\theta^\Delta(\mathbb{R}^m)^{\text{cs}}$ , and with the coarsest topology making the map  $\pi_K$  from (2.8) continuous. We will call this topology the *K-topology*.

**Step 3:** It follows formally from the previous step that the identity map

$$\Psi_\theta^\Delta(U)^L \rightarrow \Psi_\theta^\Delta(U)^K$$

is continuous whenever  $K \subseteq L$  are two compact subsets of  $U$ . Finally, we let  $\Psi_\theta^\Delta(U)$  have the coarsest topology finer than all of the  $K$ -topologies. In other words  $\Psi_\theta^\Delta(U)$  is the inverse limit of the  $\Psi_\theta^\Delta(U)^K$  as  $K$  varies over the compact subsets of  $U$ .

With  $\Psi_\theta^\Delta(U)$  topologized as above one may proceed as in [8, Section 2.2] to obtain direct analogues to the basic properties of  $\Psi_\theta(U)$  proven in that section. The following proposition is proven in the same way as [8, Theorem 2.7] and so we omit the proof.

**Proposition 2.9.** *Let  $U' \subseteq U$  be an open subset. The restriction map*

$$\Psi_\theta^\Delta(U) \longrightarrow \Psi_\theta^\Delta(U'), \quad (M, \ell, V) \mapsto (M \cap U', \ell|_{M \cap U'}, V|_{M \cap U'})$$

*is continuous.*

For each  $k \leq m$  we define  $\psi_\theta^\Delta(m, k) \subset \Psi_\theta^\Delta(\mathbb{R}^m)$  to be the subspace consisting of those  $(M, \ell, V)$  such that  $(M, \ell) \in \psi_\theta(m, k)$ . We define  $\Psi_\theta^\Delta(\mathbb{R}^\infty)$  and  $\psi_\theta(\infty, k)$  to be the direct limits of the above spaces, taken as  $m \rightarrow \infty$ .

**2.4. One-parameter families.** Below we describe a particular construction that we will use latter on in Sections 8.3 and 9.3. We first need to introduce one more bit of terminology:

**Definition 2.10.** A one parameter family  $(W_t, \ell_t) \in \Psi_\theta(\mathbb{R}^m)$ ,  $t \in \mathbb{R}$ , is said to be *locally generated by vector fields* if: Given  $t_0 \in \mathbb{R}$  and a compact subset  $A \subset \mathbb{R}^m$ , there exists  $\varepsilon > 0$  and a family of normal sections  $s_t \in \Gamma_c(\nu W_{t_0})$  with  $t \in (-\varepsilon, \varepsilon)$  such that,

$$W_{t+t_0} \cap A = s_t(W_{t_0}) \cap A \quad \text{and} \quad \ell_{t+t_0}|_{A \cap W_{t+t_0}} = (\ell_{t_0} \circ Ds_t)|_{A \cap W_{t+t_0}}.$$

**Remark 2.11.** We note that every one-parameter family in  $\Psi_\theta(\mathbb{R}^m)$  that we will explicitly encounter is locally generated by vector fields. In particular, it follows from the construction in [9, Sections 4 and 5] that the one parameter families from Propositions 8.1 and 9.2 are locally generated by vector fields.

**Construction 2.2.** For what follows, let

$$(2.9) \quad (W_t, \ell_t) \in \Psi_\theta(\mathbb{R}^m), \quad t \in \mathbb{R},$$

be a continuous one-parameter family of elements. Let  $U \subset \mathbb{R}^m$  be an open subset such that the family  $(W_t, \ell_t)$  is constant when restricted to  $\mathbb{R}^n \setminus U$ . Let  $W'$  denote the complement  $W_0 \setminus (W_0 \cap U)$ . For each  $t \in [0, 1]$  let

$$\beta_t : H_*^{\text{cpt}}(W') \longrightarrow H_*^{\text{cpt}}(W_t)$$

be the homomorphism induced by inclusion  $W' \hookrightarrow W_t$ . Fix a subspace

$$V \leq H_*^{\text{cpt}}(W').$$

For each  $t \in [0, 1]$  we let

$$(2.10) \quad V_t := \beta_t(V) \leq H_*^{\text{cpt}}(W_t)$$

be the subspace given by the image. Defined in this way, the tuple  $(W_t, \ell_t, V_t)$  is an element of the space  $\Psi_\theta^\Delta(\mathbb{R}^m)$  for all  $t \in \mathbb{R}$ .

**Proposition 2.12.** *Let  $(W_t, \ell_t) \in \Psi_\theta(\mathbb{R}^m)$  be the path from (2.9) and let  $V_t \leq H_*^{\text{cpt}}(W_t)$  be the family of subspaces defined exactly as in (2.10). Suppose that  $(W_t, \ell_t)$  is locally generated by vector fields. Then the correspondence  $t \mapsto (W_t, \ell_t, V_t) \in \Psi_\theta^\Delta(\mathbb{R}^m)$  defines a continuous path in  $\Psi_\theta^\Delta(\mathbb{R}^m)$ .*

*Proof.* To prove continuity of the family  $(W_t, \ell_t, V_t) \in \Psi_\theta^\Delta(\mathbb{R}^m)$ , it will suffice to prove that the family

$$(2.11) \quad (W_t, \ell_t, V_t)_K \in \Psi_\theta^\Delta(\mathbb{R}^m|K),$$

obtained by projecting  $(W_t, \ell_t, V_t)$  onto  $\Psi_\theta^\Delta(\mathbb{R}^m|K)$ , is continuous for all compact subsets  $K \subset \mathbb{R}^m$ . So, let  $K \subset \mathbb{R}^m$  be a compact subset. Let  $t_0 \in \mathbb{R}$ . Since  $(W_t, \ell_t)$  is locally generated by vector fields, there exists  $\varepsilon > 0$ , a (compact) neighborhood  $A \subset \mathbb{R}^n$  of  $K$ , and a family of sections  $s_t \in \Gamma_c(\nu W_{t_0})$  with  $t \in (-\varepsilon, \varepsilon)$  such that

$$W_{t+t_0} \cap A = s_t(W_{t_0}) \cap A, \quad \text{and}$$

$$\ell_{t+t_0}|_{A \cap W_{t+t_0}} = (\ell_{t_0} \circ Ds_t)|_{A \cap W_{t+t_0}}.$$

These equalities imply that the family of sections  $s_t$  is constantly zero on  $W' \cap A$ , where recall  $W' = W_0 \setminus (W_0 \cap U)$ . It follows that

$$(s_t)_*(V_{t_0})|_{A \cap W_{t+t_0}} = \beta_{t+t_0}(V)|_{A \cap W_{t+t_0}}$$

as subspaces of  $H_*^{\text{cpt}}(W_{t+t_0} \cap A)$ , and thus

$$(s_t(W_{t_0}), \ell_{t_0} \circ Ds_t, (s_t)_*V_{t_0})_K = (W_{t+t_0}, \ell_{t+t_0}, V_{t+t_0})_K \quad \text{for all } t \in (-\varepsilon, \varepsilon).$$

The correspondence  $t \mapsto (s_t(W_{t_0}), \ell_{t_0} \circ Ds_t, (s_t)_*V_{t_0})_K$  defines a continuous path in  $\Psi_\theta^\Delta(\mathbb{R}^n|K)$ , and thus the above equality implies that the family (2.11) is continuous at time  $t_0$ . The point  $t_0$  was chosen arbitrarily and so (2.11) is continuous.  $\square$

### 3. COBORDISM CATEGORIES OF HIGHLY CONNECTED ODD DIMENSIONAL MANIFOLDS

In this section we give a short exposition of the overall strategy of [9] and briefly explain its failure in odd dimensions. In particular, we study the relationship between the stabilized diffeomorphism group, the monoid  $\mathcal{M}_{2n+1}$  from the introduction and the cobordism category, which are analogous to the even dimensional situation, except for complications introduced by 3.11, which will be analyzed in detail elsewhere.

**3.1. Subcategories of  $\mathbf{Cob}_\theta$ .** Let us restrict attention to the dimensional case relevant to the main theorems asserted in the introduction. Let  $n \geq 0$  be an integer. Fix a tangential structure  $\theta : B \rightarrow BO(2n+1)$ . We proceed to filter the cobordism category  $\mathbf{Cob}_\theta$  by subcategories consisting of manifolds subject to certain connectivity conditions. Fix once and for all a  $2n$ -dimensional disk

$$(3.1) \quad D \subset (-\tfrac{1}{2}, 0] \times (-1, 1)^{\infty-1},$$

which near  $\{0\} \times \mathbb{R}^{\infty-1}$  agrees with  $(-1, 0] \times \partial D$ . Fix a  $\theta$ -structure  $\ell_D : TD \oplus \epsilon^1 \longrightarrow \theta^* \gamma^{2n+1}$ . Let  $\ell_{\mathbb{R} \times D}$  denote the  $\theta$ -structure on  $\mathbb{R} \times D$  induced by  $\ell_D$ .

**Definition 3.1.** We define a sequence of subcategories of  $\mathbf{Cob}_\theta$  as follows:

- (i) The topological subcategory  $\mathbf{Cob}_\theta^{\mathbf{m}} \subset \mathbf{Cob}_\theta$  has the same space of objects. For objects  $(M, \ell_M), (N, \ell_N) \in \text{Ob } \mathbf{Cob}_\theta^{\mathbf{m}}$ , the morphisms are given by those  $(t, W, \ell)$  such that the pair  $(W|_{[0, t]}, W|_t)$  is  $(n-1)$ -connected.
- (ii) The topological subcategory  $\mathbf{Cob}_\theta^D \subset \mathbf{Cob}_\theta^{\mathbf{m}}$  has as its objects those  $(M, \ell)$  such that

$$M \cap [(-1, 0] \times (-1, 1)^{\infty-1}] = D,$$

and such that the restriction of  $\ell$  to  $D$  agrees with  $\ell_D$ . Similarly, it has as its morphisms those  $(W, \ell)$  such that  $W \cap [\mathbb{R} \times (-1, 0] \times (-1, 1)^{\infty-1}] = \mathbb{R} \times D$ , and the restriction of  $\ell$  to  $\mathbb{R} \times D$  agrees with  $\ell_{\mathbb{R} \times D}$ .

- (iii) Let  $l \in \mathbb{Z}_{\geq -1}$ . The topological subcategory  $\mathbf{Cob}_\theta^l \subset \mathbf{Cob}_\theta^D$  is the full subcategory on those objects  $(M, \ell)$  such that  $M$  is  $l$ -connected.

We recall some results that were proven in [9]. The first follows by combining [9, Theorem 3.1 and Proposition 2.15].

**Theorem 3.2.** *The inclusions  $B\mathbf{Cob}_\theta^D \hookrightarrow B\mathbf{Cob}_\theta^{\mathbf{m}} \hookrightarrow B\mathbf{Cob}_\theta$  are weak homotopy equivalences.*

The next theorem follows from [9, Theorem 4.1].

**Theorem 3.3.** *Let  $l \leq n-1$  and suppose that  $\theta : B \longrightarrow BO(2n+1)$  is such that  $B$  is  $l$ -connected. Then the inclusion  $B\mathbf{Cob}_\theta^l \hookrightarrow B\mathbf{Cob}_\theta^{l-1}$  is a weak homotopy equivalence.*

Theorems 3.2 and 3.3 together imply the weak homotopy equivalences

$$B\mathbf{Cob}_\theta^{n-1} \simeq B\mathbf{Cob}_\theta \simeq \Omega^{\infty-1} \mathbf{MT}\theta$$

in the case that the space  $B$  is  $(n-1)$ -connected. In view of the next section let us already point out that Theorem 3.3 cannot be extended to yield a weak homotopy equivalence between  $B\mathbf{Cob}_\theta^n$  and  $B\mathbf{Cob}_\theta^{n-1}$  when  $B$  is  $n$ -connected, contrary to what happens in the even dimensional case (see [9, Theorem 5.3]). As we will see at the conclusion of this section (Remark 3.13), such a weak homotopy equivalence would lead to a contradiction to Ebert's vanishing result from [6]: The condition that  $l \leq n-1$  from the statement of Theorem 3.3 is absolutely essential. The failure of Theorem 3.3 in the case that  $l = n$  is the key motivation for the main definition of the paper as stated in the introduction.

**3.2. Reduction to a monoid.** We proceed to analyze the subcategory  $\mathbf{Cob}_\theta^n \subset \mathbf{Cob}_\theta$  and describe its relationship to the topological monoid  $\mathcal{M}_{2n+1}$  discussed in the introduction. Throughout this subsection we assume that the tangential structure  $\theta : B \longrightarrow BO(2n+1)$  is such that  $B$  is  $n$ -connected and we have to put one more restriction on it:



**Definition 3.4.** A tangential structure  $\theta : B \rightarrow BO(d)$  is said to be *once-stable* if there exists a map  $\bar{\theta} : \bar{B} \rightarrow BO(d+1)$  and a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\quad} & \bar{B} \\ \downarrow \theta & & \downarrow \bar{\theta} \\ BO(d) & \xrightarrow{\quad} & BO(d+1) \end{array}$$

which is homotopy cartesian, i.e. the induced map from  $B$  to the homotopy pullback is a weak homotopy equivalence. A tangential structure  $\theta$  is *weakly once-stable* if there exists such a diagram which is  $d$ -cartesian, i.e. the induced map from  $B$  to the homotopy pullback is  $d$ -connected.

In [9, Section 5.2] it is shown that weak once-stability is equivalent to reversibility: For a weakly once-stable tangential structure  $\theta$ , any  $\theta$ -cobordism from  $(M, \ell_M)$  to  $(N, \ell_N)$  can be made into a cobordism from  $(N, \ell_N)$  to  $(M, \ell_M)$ , something that is not true in general. We need to specify a particular subcategory of  $\mathbf{Cob}_\theta^n$ .

**Definition 3.5.** The topological subcategory  $\mathbf{Cob}_\theta^\emptyset \subset \mathbf{Cob}_\theta$  is defined to be the full subcategory on those  $\theta$ -manifolds  $(M, \ell)$  that are  $\theta$ -cobordant to the empty set. Similarly, we define  $\mathbf{Cob}_\theta^{n, \emptyset}$  to be the intersection  $\mathbf{Cob}_\theta^\emptyset \cap \mathbf{Cob}_\theta^n$ .

It follows directly from the discussion of weak once-stability that the subspace  $B\mathbf{Cob}_\theta^\emptyset \subset B\mathbf{Cob}_\theta$  is a single path component of  $B\mathbf{Cob}_\theta$ . The next proposition establishes the analogous property for  $B\mathbf{Cob}_\theta^{n, \emptyset} \subset B\mathbf{Cob}_\theta^n$ . This proposition will use our connectivity assumption on the space  $B$ , namely the fact that it is  $n$ -connected (though  $(n-1)$ -connectivity would actually suffice to prove it).

**Proposition 3.6.** *The object space  $\text{Ob } \mathbf{Cob}_\theta^{n, \emptyset}$  is a path component of  $\text{Ob } \mathbf{Cob}_\theta^n$ . In particular,  $B\mathbf{Cob}_\theta^{n, \emptyset}$  is also a path component of  $B\mathbf{Cob}_\theta^n$ .*

*Proof.* Note first, that  $\text{Ob } \mathbf{Cob}_\theta^{n, \emptyset}$  is clearly a union of path components of  $\text{Ob } \mathbf{Cob}_\theta^n$ , so it will suffice to show that  $\text{Ob } \mathbf{Cob}_\theta^{n, \emptyset}$  is path connected. To this end observe that since  $B$  is  $n$ -connected, it follows that for any object  $(M, \ell) \in \text{Ob } \mathbf{Cob}_\theta^{n, \emptyset}$ , the manifold  $M$  is diffeomorphic to the standard sphere  $S^{2n}$ : Indeed,  $M$  is  $n$ -connected and thus is a homotopy sphere. Let  $(W, \ell_W)$  be a  $\theta$ -null-bordism of  $(M, \ell)$ . Since  $B$  is  $n$ -connected, the manifold  $W$  is parallelizable over its  $n$ -skeleton. By the main theorem of [34], we may perform a sequence of surgeries on the interior of  $W$  so that the resulting manifold  $\widetilde{W}$  is a contractible manifold. Since  $n \geq 4$ , it follows that  $\widetilde{W}$  is diffeomorphic to  $D^{2n+1}$ , and thus  $M \cong S^{2n}$ . From this observation it follows from (2.2) that there is a weak homotopy equivalence

$$\text{Ob } \mathbf{Cob}_\theta^{n, \emptyset} \simeq \text{Bun}^\emptyset(TS^{2n} \oplus \epsilon^1, \theta^* \gamma^{2n+1}; \ell_D) // \text{Diff}(S^{2n}, D^{2n}),$$

where  $\text{Bun}^\emptyset(TS^{2n} \oplus \epsilon^1, \theta^* \gamma^{2n+1}; \ell_D)$  is the space of  $\theta$ -structures  $\ell$  on  $S^{2n}$  that agree with the structure  $\ell_D$  when restricted to a fixed disk  $D^{2n} \subset S^{2n}$ , such that  $(S^{2n}, \ell)$  is  $\theta$ -cobordant to the

empty set. We will show that  $\text{Bun}^\emptyset(TS^{2n} \oplus \epsilon^1, \theta^* \gamma^{2n+1}; \ell_D)$  is path connected. Since the space of  $\theta$ -structures on  $D^{2n+1}$  is contractible, to do this it will suffice to show that every element of  $\text{Bun}^\emptyset(TS^{2n} \oplus \epsilon^1, \theta^* \gamma^{2n+1}; \ell_D)$  is the restriction to the boundary of some  $\theta$ -structure on  $D^{2n+1}$ . To see this we need only observe that the surgeries making  $W$  into a disk can be chosen compatible with  $\theta$ . This is ensured by the discussion in [9, section 4.1]. The addendum stating that  $B\mathbf{Cob}_\theta^{n,\emptyset}$  is a path component of  $B\mathbf{Cob}_\theta^n$  now follows from the fact that  $\mathbf{Cob}_\theta^{n,\emptyset} \subseteq \mathbf{Cob}_\theta^n$  is a full subcategory.  $\square$

To proceed, fix once and for all an object  $(S, \ell_S) \in \text{Ob } \mathbf{Cob}_\theta^{n,\emptyset}$ . Since the object space  $\text{Ob } \mathbf{Cob}_\theta^{n,\emptyset}$  is path-connected the homotopy type of the endomorphism monoid  $\mathbf{Cob}_\theta^n(S, \ell_S)$  of  $(S, \ell_S)$  is independent of this choice. We claim that it is given by

$$(3.2) \quad \mathbf{Cob}_\theta^n(S, \ell_S) \simeq \coprod_W \text{BDiff}_\theta(W, D^{2n+1})$$

with union ranging over diffeomorphism classes (relative boundary) of  $(n-1)$ -connected,  $(2n+1)$ -dimensional, closed manifolds  $W$ , equipped with an embedding  $D^{2n+1} \hookrightarrow W$ . Composition of cobordisms obviously corresponds to connected sum of diffeomorphisms at the fixed disk. To see this let  $(t, W, \ell) \in \mathbf{Cob}_\theta^n(S, \ell_S)$ . By Definition 3.1, the manifold  $W|_{[0,t]}$  is  $(n-1)$ -connected, contains the “strip”  $[0, t] \times D \subset [0, t] \times \mathbb{R}^\infty$ , and has its boundary given by  $(\{0\} \times S) \sqcup (\{t\} \times S)$ . Let  $W'$  be the manifold obtained by removing  $[0, t] \times \text{Int}(D)$  from  $W|_{[0,t]}$ . After rescaling the length of the interval  $[0, t]$ , the boundary of  $W'$  is given by the manifold

$$K(S) := (\{0\} \times S \setminus D) \cup ([0, 1] \times \partial D) \cup (\{1\} \times S \setminus D).$$

By choosing once and for all a diffeomorphism  $K(S) \cong S^{2n}$ , elements of  $\mathcal{M}_\theta$  can be viewed as  $(n-1)$ -connected,  $(2n+1)$ -dimensional, compact manifolds with boundary identified with  $S^{2n}$ . The claim now follows from the description of the morphism spaces in (2.3). In view of this observation we set:

**Definition 3.7.** Let  $\mathcal{M}_\theta \subset \mathbf{Cob}_\theta^{n,\emptyset}$  denote the endomorphism monoid on the object  $(S, \ell_S)$ .

The result below is obtained by running the same argument as in [1, Lemma 7.3] or [9, Section 7.1] using the fact (proven above) that  $\text{Ob } \mathbf{Cob}_\theta^{n,\emptyset}$  is path connected.

**Proposition 3.8.** *The inclusion  $B\mathcal{M}_\theta \hookrightarrow B\mathbf{Cob}_\theta^{n,\emptyset}$  is a weak homotopy equivalence.*

The next proposition establishes an important property of the monoid  $\mathcal{M}_\theta$ . It is proven by following the exact same construction carried out in [8, Page 46] and [8, Proposition 4.27] in particular.

**Proposition 3.9.** *The topological monoid  $\mathcal{M}_\theta$  is homotopy commutative.*

**Remark 3.10.** For a particular choice of tangential structure we can apply the above results to the monoid  $\mathcal{M}_{2n+1}$  defined in the introduction. Let  $\theta^n : BO(2n+1)\langle n \rangle \longrightarrow BO(2n+1)$  denote the  $n$ -connected cover (note that we are following the convention of Galatius and Randal-Williams

for the use of  $\langle n \rangle$ ). To relate  $\mathcal{M}_{2n+1}$  to  $\mathcal{M}_{\theta^n}$  we use the fact that for any  $(n-1)$ -connected,  $(2n+1)$ -dimensional closed manifold  $W$  that admits a  $\theta^n$ -structure, the forgetful map

$$\mathrm{BDiff}_{\theta^n}(W, D^{2n+1}) \rightarrow \mathrm{BDiff}(W, D^{2n+1})$$

is a weak homotopy equivalence. This follows from the fact that for any such manifold  $W$ , the space of  $\theta^n$ -structures  $\mathrm{Bun}(TM, (\theta^n)^*\gamma^{2n+1}; \ell_{D^{2n+1}})$  is weakly contractible, which is proven by standard obstruction theory (see for example [1, Lemma 7.14]). Combining this fact with (3.2) and the definition of  $\mathcal{M}_{2n+1}$  in (1.7) we obtain the weak homotopy equivalence

$$(3.3) \quad \mathcal{M}_{\theta^n} \simeq \mathcal{M}_{2n+1}.$$

It follows that we may use these two monoids interchangeably and we may consider Definition 3.7 (for  $\theta = \theta^n$ ) to be a model for the monoid  $\mathcal{M}_{2n+1}$  discussed in the introduction.

**3.3. Group Completion.** Before proceeding further let us apply the group completion theorem of Segal and McDuff [20] to describe the relationship between the stable moduli space  $\mathbf{B}_\infty$  considered in the introduction (and recalled below) and the topological monoid  $\mathcal{M}_{2n+1}$ .

**Notation 3.1.** A manifold  $M$  is said to be  $n$ -parallelizable if it admits a  $\theta^n$ -structure. Let  $\mathcal{W}_{2n+1}$  denote the set of diffeomorphism classes of oriented,  $(n-1)$ -connected,  $(2n+1)$ -dimensional, closed, manifolds, that are  $n$ -parallelizable.

$\mathcal{W}_{2n+1}$  has the structure of a monoid with operation given by connected sum. Clearly  $\mathcal{W}_{2n+1} \cong \pi_0 \mathcal{M}_{2n+1}$ . We will need to use the following fact about the structure of the monoid  $\mathcal{W}_{2n+1}$ , which follow immediately from Wall's diffeomorphism classification of  $(n-1)$ -connected,  $(2n+1)$ -dimensional manifolds in [31]. Recall that a monoid  $M$  is said to be *finitely saturated* if the localization  $M[M^{-1}]$  can be constructed by inverting just finitely many elements of  $M$ .

**Proposition 3.11.** *The monoid  $\mathcal{W}_{2n+1}$  is countable, but not finitely generated or even finitely saturated.*

*Proof.* Recall that two oriented manifolds  $M$  and  $M'$  are said to be *almost diffeomorphic* if  $M$  is diffeomorphic to  $M' \# \Sigma$  where  $\Sigma$  is an oriented homotopy sphere. Since the set homotopy spheres in a given dimension is finite, it follows that there are only finitely many diffeomorphism types in a given almost-diffeomorphism class. In [31], Wall shows that the almost diffeomorphism class of any  $(n-1)$ -connected,  $(2n+1)$ -dimensional manifold  $M$ , is determined by a finite collection of algebraic invariants, each of which it turns out can take only countably many values. In the case that  $n$  is even and  $W \in \mathcal{W}_{2n+1}$ , these (almost) diffeomorphism invariants are given by the linking form  $b : \tau H_n(W) \otimes \tau H_n(W) \rightarrow \mathbb{Q}/\mathbb{Z}$ , and cohomology classes  $\hat{\phi} \in H^{n+1}(W; \mathbb{Z}/2)$  and  $\hat{\beta} \in H^{n+1}(W; \pi_n(SO))$  (see [31, Theorem 7]). The linking form is a nonsingular,  $(-1)^{n+1}$ -symmetric, bilinear pairing, and according to the classification in [33], there are countably many such objects

up to isomorphism. It follows that the set of almost-diffeomorphism classes (and hence the diffeomorphism classes) of elements of  $\mathcal{W}_{2n+1}$  is countably infinite.

The proof that  $\mathcal{W}_{2n+1}$  cannot be finitely saturated goes as follows. Let  $Q_{n+1}$  denote the monoid (under direct sum) of isomorphism classes of non-singular,  $(-1)^{n+1}$ -symmetric, bilinear pairings  $b : G \otimes G \rightarrow \mathbb{Q}/\mathbb{Z}$  with  $G$  a finite abelian group. The correspondence  $[W] \mapsto (\tau H_n(W), b)$  (sending a manifold to its linking form) defines a homomorphism of monoids  $h : \mathcal{W}_{2n+1} \rightarrow Q_{n+1}$ . In [31] Wall proves that any element of  $Q_{n+1}$  can be realized as the linking form of some element of  $\mathcal{W}_{2n+1}$ , thus proving that  $h$  is surjective. Now, from the main classification theorem in [33], it follows that the monoid  $Q_{n+1}$  is not finitely saturated. Indeed, it is proven in [33] that it surjects onto the monoid of isomorphism classes of finite abelian groups, and this monoid clearly is not finitely saturated (as finitely many elements can only ever account for torsion at finitely many primes). Since finite saturation is inherited by quotients the claim follows.  $\square$

By Proposition 3.9, the topological monoid  $\mathcal{M}_{2n+1}$  is homotopy commutative and thus we may apply the group completion theorem of McDuff and Segal from [20]. The main result of [20] implies that the natural map  $\mathcal{M}_{2n+1} \rightarrow \Omega B\mathcal{M}_{2n+1}$ , induces an isomorphism

$$(3.4) \quad H_*(\mathcal{M}_{2n+1}) [\pi_0(\mathcal{M}_{2n+1})^{-1}] \xrightarrow{\cong} H_*(\Omega B\mathcal{M}_{2n+1}).$$

Using the language of [26], this isomorphism may also be expressed as a certain map

$$(\mathcal{M}_{2n+1})_\infty \rightarrow \Omega B\mathcal{M}_{2n+1}$$

being a homology equivalence, where  $(\mathcal{M}_{2n+1})_\infty$  is the colimit of the direct system

$$(3.5) \quad \mathcal{M}_{2n+1} \xrightarrow{\cdot W_1} \mathcal{M}_{2n+1} \xrightarrow{\cdot W_1 \cdot W_2} \mathcal{M}_{2n+1} \xrightarrow{\cdot W_1 \cdot W_2 \cdot W_3} \mathcal{M}_{2n+1} \xrightarrow{\cdot W_1 \cdot W_2 \cdot W_3 \cdot W_4} \dots$$

where the  $W_i$  give a generating system of  $\pi_0(M)$  under saturation. Restricting to the path component of  $\mathcal{M}_{2n+1}$  corresponding to the sphere  $S^{2n+1}$  produces the direct system

$$\mathrm{BDiff}(W_1, D^{2n+1}) \longrightarrow \mathrm{BDiff}(W_1^{\#2} \# W_2, D^{2n+1}) \longrightarrow \mathrm{BDiff}(W_1^{\#3} \# W_2^{\#2} \# W_3, D^{2n+1}) \longrightarrow \dots$$

As in the introduction, we denote the colimit of this direct system by  $\mathbf{B}_\infty$ . Let  $\Omega_0 B\mathcal{M}_{2n+1} \subseteq \Omega B\mathcal{M}_{2n+1}$  denote the path-component that contains the constant loop. We obtain:

**Proposition 3.12.** *The above construction produces a homology equivalence*

$$\mathbf{B}_\infty \rightarrow \Omega_0 B\mathcal{M}_{2n+1}.$$

**Remark 3.13.** We are now in a position to prove that the inclusion  $\Omega B\mathbf{Cob}_{\theta^n}^n \hookrightarrow \Omega B\mathbf{Cob}_{\theta^n}$  cannot be a weak homotopy equivalence as discussed at the end of Subsection 3.1. Recall from 3.13 the

commutative diagram

$$(3.6) \quad \begin{array}{ccc} \Omega_0 B\mathcal{M}_{2n+1} & \longrightarrow & \Omega_0^\infty \mathbf{MT}\theta^n \\ \simeq_{H_*} \uparrow & & \uparrow \amalg \mathcal{P}_W \\ \mathbf{B}_\infty & \longleftarrow & \amalg_W \mathbf{BDiff}(W, D^{2n+1}) \end{array}$$

where the bottom-horizontal map is induced by the inclusions of the terms into their colimit and the right-vertical map is the *scanning map* (see [6]). The top arrow is induced by the composite

$$B\mathcal{M}_{2n+1} \xrightarrow{\simeq} B\mathbf{Cob}_{\theta^n}^n \longrightarrow B\mathbf{Cob}_{\theta^n} \xrightarrow{\simeq} \Omega^{\infty-1} \mathbf{MT}\theta^n.$$

Since  $H_*(\_)$  preserves direct limits, the bottom map is certainly surjective in homology, and thus  $B\mathbf{Cob}_{\theta^n}^n \longrightarrow B\mathbf{Cob}_{\theta^n}$  being a weak equivalence would imply surjectivity (in homology) of the right-vertical map. However, the main theorem of [6] implies that this map has a non-trivial kernel in cohomology, even rationally, and thus cannot be surjective in homology.

By the above remark we therefore see that the failure of Theorem 3.3 in the case that  $l = n$  is fundamental, and not merely a technical shortcoming of the methods of [9].

#### 4. COBORDISM CATEGORIES OF MANIFOLDS EQUIPPED WITH SURGERY DATA

In this section we give the main definition of the paper (Definition 4.6) and state the main technical theorems for us to prove (Theorems 4.10 - 4.13). In Subsection 4.1 we cover some preliminary definitions and terminology regarding bilinear and quadratic forms and their Lagrangian subspaces.

**4.1. Preliminaries on quadratic forms and Lagrangian subspaces.** Let  $\varepsilon = \pm 1$ . An  $\varepsilon$ -symmetric *bilinear form* is a pair  $(\mathbf{M}, \lambda)$  where  $\mathbf{M}$  is a finitely generated  $\mathbb{Z}$ -module and  $\lambda : \mathbf{M} \otimes \mathbf{M} \longrightarrow \mathbb{Z}$  is a bilinear map with the property that  $\lambda(x, y) = \varepsilon \cdot \lambda(y, x)$  for all  $x, y \in \mathbf{M}$ . An  $\varepsilon$ -symmetric bilinear form  $(\mathbf{M}, \lambda)$  is said to be *non-singular* if the map

$$\mathbf{M} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbf{M}, \mathbb{Z}), \quad x \mapsto \lambda(x, \_)$$

becomes an isomorphism after modding out torsion. If  $\mathbf{V} \leq \mathbf{M}$  is a submodule, we let  $\mathbf{V}^\perp$  denote the orthogonal complement of  $\mathbf{V}$  in  $\mathbf{M}$ , i.e.  $\mathbf{V}^\perp = \{x \in \mathbf{M} \mid \lambda(x, v) = 0 \text{ for all } v \in \mathbf{V}\}$ . An  $\varepsilon$ -*quadratic form* is a triple  $(\mathbf{M}, \lambda, \mu)$ , such that  $(\mathbf{M}, \lambda)$  is an  $\varepsilon$ -symmetric form and  $\mu \rightarrow \mathbb{Z}/(1 - \varepsilon)$  is a quadratic refinement of  $\lambda$  in the sense that it is quadratic and satisfies

$$\mu(p + q) = \mu(p) + \mu(q) + [\lambda(p, q)]$$

**Definition 4.1.** Let  $\varepsilon = \pm 1$ , and let  $(\mathbf{M}, \lambda)$  be an  $\varepsilon$ -quadratic form. A submodule  $\mathbf{L} \leq \mathbf{M}$  will be called a *lagrangian* if  $\mathbf{L} = \mathbf{L}^\perp$  and  $\mu|_{\mathbf{L}} = 0$ .

**Remark 4.2.** Usually the definition of a lagrangian is only uttered for non-singular quadratic forms whose underlying module is torsion free. For us, however, this would be an inconvenient restriction later on.

**Remark 4.3.** Let us note immediately, that a symmetric form (i.e.  $\varepsilon = 1$ ) admits a quadratic refinement, only if it is even in the sense that  $\lambda(p, p)$  is even for every  $p \in \mathbf{M}$ . If that is the case, then there exists a unique refinement, namely  $\mu(p) = \lambda(p, p)/2$ . In particular the second condition in the definition of lagrangian is implied by the first in this case and one can wholly disregard the quadratic refinement.

In the case of an anti-symmetric form the situation is quite the opposite: Any torsionfree such form admits a subspace  $L$  with  $L^\perp = L$ , and thus admits a quadratic refinement (since it can then be split apart into standard hyperbolics all of which do admit a quadratic refinement), but such a quadratic refinement is neither unique nor forced to vanish on  $L$ .

Let  $M$  be a  $2n$ -dimensional compact manifold. The main example of a bilinear form that we will consider is the intersection pairing

$$\lambda : H_n(M) \otimes H_n(M) \longrightarrow \mathbb{Z}, \quad (x, y) \mapsto \langle D(x), j(y) \rangle,$$

where  $D : H_n(M) \xrightarrow{\cong} H^n(M, \partial M)$  is the Lefschetz duality isomorphism,  $j : H_n(M) \longrightarrow H_n(M, \partial M)$  is the map induced by inclusion, and  $\langle \cdot, \cdot \rangle : H^n(M, \partial M) \otimes H_n(M, \partial M) \longrightarrow \mathbb{Z}$  is the pairing between homology and cohomology. It follows from basic properties of the Lefschetz-duality isomorphism that  $\lambda$  is  $(-1)^n$ -symmetric. In the case that  $M$  is a closed manifold it follows that the form  $(H_n(M), \lambda)$  is non-singular. Throughout the paper we will refer to this bilinear form  $(H_n(M), \lambda)$  as the *intersection form* associated to  $M$  and we will now endow it with a quadratic refinement when  $M$  comes equipped with a highly connected tangential structure. With  $\dim(M) = 2n$ , the construction of the quadratic refinement breaks down in two separate cases depending on the parity of the integer  $n$ . We outline the constructions of the quadratic refinement below.

**Construction 4.1.** Suppose  $n$  even and consider a  $2n$ -dimensional manifold  $M$  equipped with a tangential structure  $\theta : B \longrightarrow BO(2n + 1)$ , with  $B$   $n$ -connected. It follows immediately that  $v_n(TM) = 0$  where  $v_n$  denotes the  $n$ -th Wu class. For even  $n$ , the element  $v_n(TM)$  is characteristic for the modulo-2 intersection pairing by the Wu formula, and thus its vanishing forces this pairing to be even. From Remark 4.3, we automatically obtain a quadratic refinement of the intersection form for such manifolds.

In the case of odd  $n$  one has to work harder to obtain a quadratic refinement for the intersection form of a  $2n$ -dimensional manifold. We can only produce such a refinement in the cases  $n = 5$  or  $n \geq 9$ . If  $\theta : B \longrightarrow BO(2n + 1)$  is a tangential structure with  $B$   $n$ -connected, it follows that the  $(n + 1)$ st Wu-class  $v_{n+1}(-\theta^*\gamma^{2n+1})$  vanishes by a calculation of Stong [29]. For  $n = 5$  this is immediate. For  $n \geq 9$  Stong's calculations imply that  $v_{n+1} \in H^{n+1}(BO\langle n - 1 \rangle, \mathbb{Z}/2)$  lies in

the image of the lowest Postnikov section, which implies that it vanishes in  $H^{n+1}(BO\langle n \rangle, \mathbb{Z}/2)$  through which the classifying map of  $\theta^*\gamma^{2n+1}$  factors. But more is true: One obtains a canonical Wu orientation, i.e. a lift of the classifying map of the normal bundle of  $M$  to the homotopy fibre of  $v_{n+1}(-\theta^*\gamma^{2n+1}) : B \rightarrow K(\mathbb{Z}/2, n+1)$ , since such lifts are parametrised by  $H^n(B, \mathbb{Z}/2) = 0$  once the obstruction  $v_{n+1}(-\theta^*\gamma^{2n+1})$ -vanishes.

In his work on the Arf-Kervaire invariant Browder showed that such a Wu orientation on the normal bundle of a manifold induces a quadratic refinement of its intersection pairing. Later Brown and Jones gave a different construction of this refinement and we will recall it in our situation (compare e.g. [15]), since we shall need a few of its properties:

**Construction 4.2.** Let  $n$  be odd with either  $n = 5$  or  $n \geq 9$ . Let  $\theta : B \rightarrow BO(2n+1)$  be such that  $B$  is  $n$ -connected and let  $(M, \ell)$  be a  $2n$ -dimensional  $\theta$ -manifold. By the discussion above, the stable normal bundle of  $M$  has a canonical Wu-orientation.

Given a class  $x \in H_n(M, \mathbb{Z}/2)$  we can represent its Poincaré dual by a map  $(M, \partial M) \rightarrow (K(\mathbb{Z}/2, n), pt)$ . Since  $B$  is  $n$ -connected by assumption, this map determines an element  $\mu(x)$  in the bordism group  $\Omega_{2n}^{(n)}(K(\mathbb{Z}/2, n), pt)$ . With our assumptions on the integer  $n$ , it turns out that this bordism group is isomorphic to  $\mathbb{Z}/2$ . By identifying  $\Omega_{2n}^{(n)}(K(\mathbb{Z}/2, n), pt) \cong \mathbb{Z}/2$ , the assignment  $x \mapsto \mu(x)$  yields the desired quadratic refinement of the intersection form. The argument that  $\Omega_{2n}^{(n)}(K(\mathbb{Z}/2, n), pt) \cong \mathbb{Z}/2$  proceeds as follows:

First observe that the map  $\Sigma^\infty K(\mathbb{Z}/2, n) \rightarrow H(\mathbb{Z}/2, n)$  admits a lift into the homotopy fibre  $F$  of

$$Sq^{n+1} : H(\mathbb{Z}/2, n) \rightarrow H(\mathbb{Z}/2, 2n+1).$$

Any such lift turns out to be a  $(2n+1)$ -equivalence by a direct calculation of the cohomology groups involved. In particular,  $\Omega_{2n}^{(n)}(K(\mathbb{Z}/2, n), pt) \cong \Omega_{2n}^{(n)}(F)$ . The fibre sequence leads to the exact sequence

$$H_{n+1}(MO\langle n \rangle, \mathbb{Z}/2) \xrightarrow{Sq_*^{n+1}} H_0(MO\langle n \rangle, \mathbb{Z}/2) \rightarrow \Omega_{2n}^{(n)}(F) \rightarrow H_n(MO\langle n \rangle, \mathbb{Z}/2).$$

Clearly the fourth term is 0 and the second one  $\mathbb{Z}/2$ . The first map may be identified with

$$\chi(Sq^{n+1})^* : H^{n+1}(MO\langle n \rangle, \mathbb{Z}/2)^* \rightarrow H^0(MO\langle n \rangle, \mathbb{Z}/2)^*$$

Since  $\chi(Sq^{n+1})(u) = v_{n+1}$  by definition of the Wu-class, the map vanishes and we obtain the desired isomorphism.

The two most important facts about this refinement for us are that 1) it is preserved under codimension zero embeddings and 2) on an  $(n-1)$ -connected,  $2n$ -dimensional manifold it agrees with Wall's self-intersection form, see [35, Theorem 5.2] or [34]: The selfintersection pairing is defined on the group of regular homotopy classes of immersions  $S_n(M)$  and given a tangential structure map  $l : M \rightarrow B$  Wall in [35] (see [4, Lemma 4.60] for a pleasant exposition) constructs a map



$\pi_{n+1}(l) \rightarrow S_n(M)$ . For  $n$ -connected  $B$  we obtain a surjection  $\pi_{n+1}(l) \rightarrow H_n(M)$  from the exact sequence of  $l$  and the claim is that the two arising compositions

$$\pi_{n+1}(l) \longrightarrow H_n(M) \longrightarrow \mathbb{Z}/2 \quad \text{and} \quad \pi_{n+1}(l) \longrightarrow S_n(M) \longrightarrow \mathbb{Z}/2$$

agree when  $B$  is  $n$ -connected. In his widely circulated thesis Jones showed that his definition agrees with Browder's (this is also discussed in [3]). From here one can either use Browder's proof [2, IV.4.7] that his pairing precisely obstructs finding embedded spheres with trivial normal bundle as does the selfintersection pairing (though one has to remove the assumption of the target being a Poincaré complex) or one can use Ranicki's [27, Proposition 5.2 & 5.3] and the discussion thereafter which directly shows that both pairings are equal to one defined via Wu classes in chain complexes. For the reader's convenience we supply a proof of the first fact:

**Proposition 4.4.** *For an embedding  $i : W \rightarrow M$  for a closed  $2n$ -manifold  $M$  and a compact  $2n$ -manifold  $W$ , the map  $i_* : H_n(W) \rightarrow H_n(M)$  preserves both the intersection and the selfintersection pairing.*

*Proof.* That the intersection pairing is preserved is a simple calculation:

$$\begin{aligned} \lambda_M(i_*(x), i_*(y)) &= \langle i_*(x), D^M i_*(y) \rangle_M \\ &= \langle i_*(x), i^! D^{(W, \partial W)} y \rangle_M \\ &= \langle i_! i_*(x), D^{(W, \partial W)} y \rangle_{(W, \partial W)} \\ &= \langle incl_*(x), D^{(W, \partial W)} y \rangle_{(W, \partial W)} \\ &= \lambda_W(x, y) \end{aligned}$$

Where the shriek maps are induced by  $(M, \emptyset) \rightarrow (M, M - \mathring{W}) \leftarrow (W, \partial W)$  (the right arrow inducing an isomorphism by excision),  $incl$  denotes the inclusion  $(W, \emptyset) \rightarrow (W, \partial W)$  and  $D^M i_*(y) = i^! D^{(W, \partial W)} y$  follows from the naturality of cap products  $H_*(X, A \cup B) \times H^*(X, A) \rightarrow H^*(X, B)$  applied to  $(M, \emptyset, \emptyset) \rightarrow (M, M - \mathring{W}, \emptyset)$  using the fact that  $i_!([M]) = [W, \partial W]$ .

The preservation of the selfintersection pairing now follows since also in  $\Omega^{(n)}$  we have  $i_!([M]) = [W, \partial W]$  (since this can be checked locally around some point, by the definition of fundamental classes) so

$$\begin{aligned} \mu_M(i_*(y)) &= [D^M i_*(y) : (M, \emptyset) \rightarrow (K(\mathbb{Z}, n), pt)] \\ &= [i^! D^{(W, \partial W)} y : (M, \emptyset) \rightarrow (K(\mathbb{Z}, n), pt)] \\ &= (D^{(W, \partial W)} y)_* i_!([M]) \\ &= (D^{(W, \partial W)} y)_*([W, \partial W]) \\ &= [D^{(W, \partial W)} y : (W, \partial W) \rightarrow (K(\mathbb{Z}, n), pt)] \\ &= \mu_{(W, \partial W)}(y) \end{aligned}$$

□

Lagrangian subspaces of the intersection form play an important role in surgery theory. The theorem below is the simply connected case of Wall's embedding theorem (see e.g. [27, Proposition 5.2] or [4, Proposition 4.13]) and is essentially what motivates our use of Lagrangian subspaces in the definition of the cobordism category  $\mathbf{Cob}_\theta^{\mathcal{L}}$  in the next section. It will play a key role in the proofs of the main results of this paper (namely Theorem 4.12).

**Theorem 4.5.** *Let  $n \geq 3$  and let  $\theta : B \rightarrow BO(2n+1)$  be such that  $B$  is  $n$ -connected. Let  $(M, \ell)$  be an  $(n-1)$ -connected,  $2n$ -dimensional, closed,  $\theta$ -manifold. Let  $L \leq H_n(M)$  be a Lagrangian subspace. Then there exists a finite set  $\Sigma$  and an embedding  $f : \Sigma \times S^n \times D^n \rightarrow M$  that satisfies the following conditions:*

- (a) *The  $\theta$ -structure on  $\Sigma \times S^n \times D^n$  given by the composition*

$$T(\Sigma \times S^n \times D^n) \oplus \epsilon^1 \xrightarrow{Df \oplus Id} TM \oplus \epsilon^1 \xrightarrow{\ell} \theta^* \gamma^{2n+1},$$

*extends to a  $\theta$ -structure on  $\Sigma \times D^{n+1} \times D^n$ .*

- (b) *The homology classes*

$$[f|_{\{\sigma\} \times S^n \times \{0\}}] \in H_n(M), \quad \sigma \in \Sigma,$$

*yield a basis for the Lagrangian subspace  $L \leq H_n(M)$ .*

Note that the assumptions on  $M$  automatically force  $H_n(M)$  to be torsion-free and thus make the lagrangian a free module as well, so that (b) indeed makes sense. Condition (b) in particular implies that the manifold  $\widetilde{M}$  obtained by performing surgery on the embedding  $f$  is  $n$ -connected, i.e. a homotopy sphere.

**4.2. Cobordism categories of manifolds equipped with Lagrangian subspaces.** Fix a tangential structure  $\theta : B \rightarrow BO(2n+1)$ . Recall the topological subcategory  $\mathbf{Cob}_\theta^{\mathbf{m}} \subset \mathbf{Cob}_\theta$  from Definition 3.1, whose morphisms consist of cobordisms that are  $(n-1)$ -connected relative to their outgoing boundary.

**Definition 4.6.** The non-unital topological category  $\mathbf{Cob}_\theta^{\mathcal{L}}$  has as its object space the subspace of  $\psi_\theta^\Delta(\infty, 0)$  given by those tuples  $(M, \ell, L)$  for which  $L \leq H_n(M)$  is a Lagrangian subspace with respect to the intersection form  $(H_n(M), \lambda, \mu)$ . The space of morphisms is given by the following subspace of the product

$$\mathbb{R} \times \psi_\theta(1 + \infty, 1) \times \psi_\theta^\Delta(\infty, 0) \times \psi_\theta^\Delta(\infty, 0) :$$

A tuple  $(t, (W, \ell), (M, \ell_M, L_M), (N, \ell_N, L_N))$  is a morphism (from  $(M, \ell_M, L_M)$  to  $(N, \ell_N, L_N)$ ) if

- (i)  $(t, W, \ell) \in \mathbf{Cob}_\theta^{\mathbf{m}}((M, \ell_M), (N, \ell_N))$
- (ii)  $\iota_{in}(L_M) = \iota_{out}(L_N)$  as subspaces of  $H_n(W)$ .

Here,  $\iota_{\text{in}} : H_n(M) \rightarrow H_n(W)$  and  $\iota_{\text{out}} : H_n(N) \rightarrow H_n(W)$  are the homomorphisms induced by the boundary inclusions  $M = W|_0 \hookrightarrow W|_{[0,t]} \hookleftarrow W|_t = N$ .

**Remark 4.7.** We note that the forgetful functor  $\mathbf{Cob}_\theta^\mathcal{L} \rightarrow \mathbf{Cob}_\theta$  is faithful. It is, however, the increase in extra structure on objects that makes the definition of the entire morphism space of  $\mathbf{Cob}_\theta^\mathcal{L}$  more complicated than that of  $\mathbf{Cob}_\theta$  as a cobordism no longer determines its source or target.

**Notation 4.1.** When denoting a morphism in  $\mathbf{Cob}_\theta^\mathcal{L}$  we will nevertheless usually drop the source and target from the notation and write  $(t, W, \ell) := (t, (W, \ell), (M, \ell_M, L_M), (N, \ell_N, L_N))$ .

We proceed to filter the cobordism category  $\mathbf{Cob}_\theta^\mathcal{L}$  by subcategories analogous to those from Definition 3.1. Let  $D \subset (-\frac{1}{2}, 0] \times (-1, 1)^{\infty-1}$  be the  $2n$ -dimensional disk from (3.1). Let  $\ell_D$  be the chosen  $\theta$ -structure on  $D$  and let  $\ell_{\mathbb{R} \times D}$  be the  $\theta$ -structure on  $\mathbb{R} \times D$  induced by  $\ell_D$ .

**Definition 4.8.** We define a sequence of subcategories of  $\mathbf{Cob}_\theta^\mathcal{L}$  as follows:

- (i) The topological subcategory  $\mathbf{Cob}_\theta^{\mathcal{L}, D} \subseteq \mathbf{Cob}_\theta^\mathcal{L}$  has as its objects those  $(M, \ell, L)$  with  $(M, \ell) \in \text{Ob } \mathbf{Cob}_\theta^D$ . Similarly, it has as its morphisms those  $(t, W, \ell)$  such that

$$W \cap [\mathbb{R} \times (-1, 0] \times (-1, 1)^{\infty-1} = \mathbb{R} \times D,$$

and the restriction of  $\ell$  to  $\mathbb{R} \times D$  agrees with  $\ell_{\mathbb{R} \times D}$ . In other words  $(t, W, \ell)$  is a morphism in  $\mathbf{Cob}_\theta^D$ .

- (ii) Let  $l \in \mathbb{Z}_{\geq -1}$ . The topological subcategory  $\mathbf{Cob}_\theta^{\mathcal{L}, l} \subseteq \mathbf{Cob}_\theta^{\mathcal{L}, D}$  is the full-subcategory on those objects  $(M, \ell, L)$  such that  $M$  is  $l$ -connected. In other words  $(M, \ell) \in \text{Ob } \mathbf{Cob}_\theta^l$ .

**Remark 4.9.** Let  $(M, \ell, L) \in \text{Ob } \mathbf{Cob}_\theta^{\mathcal{L}, n}$ . By definition,  $M$  is  $n$ -connected and thus automatically  $L = 0$ . Therefore the forgetful functor  $\mathbf{Cob}_\theta^{\mathcal{L}, n} \rightarrow \mathbf{Cob}_\theta^n$  is an isomorphism, and  $\mathbf{Cob}_\theta^{\mathcal{L}, n}$  can be considered a subcategory of  $\mathbf{Cob}_\theta$ .

**4.3. The main theorems.** We now state a series of theorems that we will prove in the sections to come. Supposing them for a moment we will then deduce all claims made in the introduction from them. The first theorem is proven using the same technique as [9, Corollary 2.17]. We give a sketch of the proof in Section 6.5.

**Theorem 4.10.** *The inclusion  $B\mathbf{Cob}_\theta^{\mathcal{L}, D} \hookrightarrow B\mathbf{Cob}_\theta^\mathcal{L}$  is a weak homotopy equivalence.*

The next theorem is proven in Section 8. This is the first theorem of the paper whose proof requires a substantial amount of technical work. The constructions that go into it, however, closely resemble those from [9].

**Theorem 4.11.** *Let  $l \leq n-1$  and suppose that  $\theta : B \rightarrow BO(2n+1)$  is such that  $B$  is  $l$ -connected. Then the inclusion  $B\mathbf{Cob}_\theta^{\mathcal{L}, l} \hookrightarrow B\mathbf{Cob}_\theta^{\mathcal{L}, l-1}$  is a weak homotopy equivalence.*

By combining the theorems stated above, we obtain the weak homotopy equivalence

$$BCob_{\theta}^{\mathcal{L}, n-1} \simeq BCob_{\theta}^{\mathcal{L}}$$

for every  $\theta$  with  $B$   $(n-1)$ -connected.

We now consider the subcategory  $Cob_{\theta}^{\mathcal{L}, n} \subset Cob_{\theta}^{\mathcal{L}}$ .

**Theorem 4.12.** *Let  $n \geq 4$  be an integer except 7. Suppose that  $\theta : B \rightarrow BO(2n+1)$  is weakly once stable and that  $B$  is  $n$ -connected. Then the inclusion  $BCob_{\theta}^{\mathcal{L}, n} \hookrightarrow BCob_{\theta}^{\mathcal{L}, n-1}$  is a weak homotopy equivalence.*

We again emphasize that this theorem is in stark contrast to the situation for cobordism categories without Lagrangians. By the isomorphism  $Cob_{\theta}^{\mathcal{L}, n} \cong Cob_{\theta}^n$  discussed in Remark 4.9, the above theorems imply that there is a weak homotopy equivalence  $BCob_{\theta}^n \simeq BCob_{\theta}^{\mathcal{L}}$  whenever  $\theta$  and  $n$  satisfy the conditions of Theorem 4.12. Finally we prove in Section 5:

**Theorem 4.13.** *The operation of disjoint union gives  $BCob_{\theta}^{\mathcal{L}}$  the structure of a special  $\Gamma$ -space. In particular,  $BCob_{\theta}^{\mathcal{L}, \emptyset}$  carries the structure of an infinite loop space.*

**Remark 4.14.** One may wonder whether  $BCob_{\theta}^{\mathcal{L}}$  is a *very special*  $\Gamma$ -space. This is indeed the case, but a proof is most readily given by showing  $Cob_{\theta}^{\mathcal{L}}$  to be equivalent to a cobordism category with no connectivity assumption on the morphisms, where it is then immediate that the components form a group. Since we will not make use of this more general assertion we have omitted it.

Supposing the above theorems, we now proceed to prove all results stated in the introduction. Let  $n \geq 4$  be an integer except 7. The  $n$ -connected cover  $BO(2n+1)\langle n \rangle \rightarrow BO(2n+1)$  clearly satisfies the conditions stated in Theorem 4.12. Theorem A, follows immediately by combining the weak homotopy equivalence  $BCob_{\theta^n}^n \simeq BCob_{\theta^n}^{\mathcal{L}}$  with the weak homotopy equivalences

$$BM_{2n+1} \simeq BM_{\theta^n} \simeq BCob_{\theta^n}^{n, \emptyset},$$

proven in Section 3.2. Corollary B then follows by combining this with Theorem 4.13. Corollary C follows by combining the weak homotopy equivalence  $BM_{2n+1} \simeq BCob_{\theta^n}^{\mathcal{L}, \emptyset}$ , with the homology equivalence  $\mathbf{B}_{\infty} \rightarrow \Omega_0 BM_{2n+1}$  established in Section 3.3. To obtain Corollary D we do the following. Consider the commutative diagram

$$\begin{array}{ccc} \coprod_W B\text{Diff}(W, D^{2n+1}) & \longrightarrow & \Omega_0 BCob_{\theta^n} \\ \downarrow & & \uparrow \\ \mathbf{B}_{\infty} & \xrightarrow{\simeq_{H*}} & \Omega_0 BCob_{\theta^n}^{\mathcal{L}} \end{array}$$

and notice that the downwards left arrow (which is given by the inclusions of the terms of the colimit sequence into the colimit) is surjective on homology and thus injective in cohomology (with rational coefficients). The kernel of the left hand vertical map in cohomology is therefore the same as that of the top horizontal arrow. Composing with the weak homotopy equivalence  $\Omega BCob_{\theta^n} \xrightarrow{\simeq} \Omega^{\infty} MT\theta^n$

yields the claim. With the main results from the introduction established, the rest of the paper is devoted to proving Theorems 4.10 - 4.13.

## 5. INFINITE LOOPSPACES

In this section we prove Theorem 4.13 which asserts that  $B\mathbf{Cob}_\theta^{\mathcal{L},\emptyset}$  is an infinite loop space. We do this by constructing a  $\Gamma$ -space structure on the semi-simplicial nerve of the category  $\mathbf{Cob}_\theta^{\mathcal{L},\emptyset}$ . In order to do this we first need to introduce a convenient model for the nerve of  $\mathbf{Cob}_\theta^{\mathcal{L},\emptyset}$ .

**5.1. A model for the nerve.** We begin by defining a semi-simplicial space equivalent to the nerve of  $\mathbf{Cob}_\theta$ . Let us fix some notation. For  $k \in \mathbb{Z}$ , we let  $\mathbb{R}_{\text{ord}}^k \subset \mathbb{R}^k$  denote the subspace consisting of those tuples  $(x_0, \dots, x_{k-1})$  with  $x_0 < x_1 < \dots < x_{k-1}$ . We denote by  $\mathbb{R}_{>0}^k \subset \mathbb{R}^k$  the subspace consisting of those tuples  $(x_0, \dots, x_{k-1})$  with  $x_i > 0$  for all  $i = 0, \dots, k-1$ . Let  $\theta : B \rightarrow BO(d)$  be as above in the definition of  $\mathbf{Cob}_\theta$

**Definition 5.1.** For each  $p \in \mathbb{Z}_{\geq 0}$ ,  $\mathbf{C}_p$  is defined to be the space of tuples  $(a, \varepsilon, W, \ell)$  with:

- $a = (a_0, \dots, a_p) \in \mathbb{R}_{\text{ord}}^{p+1}$ ,
- $\varepsilon = (\varepsilon_0, \dots, \varepsilon_p) \in \mathbb{R}_{>0}^{p+1}$ ,
- $(W, \ell) \in \psi_\theta(\infty, 1)$ ,

subject to the following conditions: Over each interval  $(a_i - \varepsilon_i, a_i + \varepsilon_i)$ , the  $\theta$ -manifold  $(W, \ell)$  is cylindrical. By this we mean for each  $i = 0, \dots, p$ ,

$$W|_{(a_i - \varepsilon_i, a_i + \varepsilon_i)} = (a_i - \varepsilon_i, a_i + \varepsilon_i) \times W|_{a_i},$$

as  $\theta$ -manifolds, where  $(a_i - \varepsilon_i, a_i + \varepsilon_i) \times W|_{a_i}$  is equipped with the  $\theta$ -structure induced by  $\ell|_{a_i}$  on  $W|_{a_i}$ . Each  $\mathbf{C}_p$  is topologized as a subspace of  $\mathbb{R}_{\text{ord}}^{p+1} \times \mathbb{R}_{>0}^{p+1} \times \psi_\theta(\infty, 1)$ . The assignment  $[p] \mapsto \mathbf{C}_p$  yields a semi-simplicial space  $\mathbf{C}_\bullet$  with face maps defined as follows: For each  $i = 0, \dots, p$ , the face map  $d_i : \mathbf{C}_p \rightarrow \mathbf{C}_{p-1}$  is defined by the formula  $d_i(a, \varepsilon, W, \ell) = (a(i), \varepsilon(i), W, \ell)$ , where  $a(i)$  is the  $(p-1)$ -tuple obtained from  $a$  by removing the  $i$ -th term, and  $\varepsilon(i)$  is defined similarly. The assignment  $[p] \mapsto \mathbf{C}_p$  makes  $\mathbf{C}_\bullet$  into a semi-simplicial space.

For each  $p$ -simplex  $(a, \varepsilon, W, \ell) \in \mathbf{C}_p$ , the tuple  $(a, W, \ell)$  determines a unique element in  $N_p \mathbf{Cob}_\theta$  by remembering only the various  $W|_{[a_i, a_{i+1}]}$  (appropriately translated) with their induced theta structures. This correspondence defines a semi-simplicial map  $\mathbf{C}_\bullet \rightarrow N_\bullet \mathbf{Cob}_\theta$ . This map is easily shown to be a level-wise weak homotopy equivalence, and it induces a weak homotopy equivalence  $|\mathbf{C}_\bullet| \simeq B\mathbf{Cob}_\theta$  (see [8, Theorem 3.9] for the proof of this weak homotopy equivalence, and for detailed definition of the map). Using  $\mathbf{C}_\bullet$ , we proceed to define a model for the nerve of  $\mathbf{Cob}_\theta^{\mathcal{L}}$ .

**Definition 5.2.** For each  $p \in \mathbb{Z}_{\geq 0}$ ,  $\mathbf{C}_p^{\mathcal{L}}$  is the space of tuples  $(a, \varepsilon, W, \ell, L)$  subject to the following conditions:

- (a) The tuple  $(a, \varepsilon, W, \ell)$  is an element of  $\mathbf{C}_p$ . Furthermore, for each  $i = 0, \dots, p$ , the pair  $(W|_{[a_i, a_{i+1}]}, W|_{a_{i+1}})$  is  $(n-1)$ -connected (and thus  $W|_{[a_i, a_{i+1}]}$  determines a morphism in the category  $\mathbf{Cob}_\theta^{\mathbf{m}}$ ),
- (b)  $L = (L_0, \dots, L_p)$  is a  $(p+1)$ -tuple with  $L_i \leq H_n(W|_{a_i})$  a Lagrangian subspace for each  $i = 0, \dots, p$ .
- (c) For each  $i = 0, \dots, p$ , we require  $\iota_i^{\text{in}}(L_i) = \iota_{i+1}^{\text{out}}(L_{i+1})$  where

$$\iota_i^{\text{in}} : H_n(W|_{a_i}) \longrightarrow H_n(W|_{[a_i, a_{i+1}]}) \quad \text{and} \quad \iota_{i+1}^{\text{out}} : H_n(W|_{a_{i+1}}) \longrightarrow H_n(W|_{[a_i, a_{i+1}]})$$

are the maps induced by inclusion.

Each space  $\mathbf{C}_p^\mathcal{L}$  is topologized as a subspace of the product  $\mathbf{C}_p \times (\text{Ob } \mathbf{Cob}_\theta^\mathcal{L})^{\times(p+1)}$ . The assignment  $[p] \mapsto \mathbf{C}_p^\mathcal{L}$  defines a semi-simplicial space. For all  $i = 0, \dots, p$ , the face map  $d_i : \mathbf{C}_p^\mathcal{L} \longrightarrow \mathbf{C}_{p-1}^\mathcal{L}$  defined by  $d_i(a, \varepsilon, W, \ell, L) = (a(i), \varepsilon(i), W, \ell, L(i))$ , where  $L(i)$  is the  $(p-1)$ -tuple defined by removing the  $i$ th term from  $L$ ;  $a(i)$  and  $\varepsilon(i)$  are defined similarly. As in the case with  $\mathbf{C}_\bullet$ , there is a semi-simplicial map  $\mathbf{C}_\bullet^\mathcal{L} \longrightarrow N_\bullet \mathbf{Cob}_\theta^\mathcal{L}$ . Applying the same argument as in [8, Theorem 3.9], it follows that this semi-simplicial map is a level-wise weak homotopy equivalence, and thus induces the weak homotopy equivalence  $|\mathbf{C}_\bullet^\mathcal{L}| \simeq B\mathbf{Cob}_\theta^\mathcal{L}$ . Corresponding to the sequence of subcategories from Definition 4.8, we have a sequence of sub-semi-simplicial spaces

$$(5.1) \quad \mathbf{C}_\bullet^{\mathcal{L}, l} \subset \dots \subset \mathbf{C}_\bullet^{\mathcal{L}, -1} \subset \mathbf{C}_\bullet^{\mathcal{L}, D} \subset \mathbf{C}_\bullet^\mathcal{L},$$

defined analogously to the subcategories of Definition 4.8. Applying the construction from [8, Theorem 3.9] again, we obtain weak homotopy equivalences

$$|\mathbf{C}_\bullet^\mathcal{L}| \simeq B\mathbf{Cob}_\theta^\mathcal{L}, \quad |\mathbf{C}_\bullet^{\mathcal{L}, D}| \simeq B\mathbf{Cob}_\theta^{\mathcal{L}, D}, \quad |\mathbf{C}_\bullet^{\mathcal{L}, l}| \simeq B\mathbf{Cob}_\theta^{\mathcal{L}, l}.$$

**5.2.  $\Gamma$ -spaces.** We now proceed to show that  $|\mathbf{C}_\bullet^\mathcal{L}|$  from the previous subsection has the structure of a  $\Gamma$ -space. We quickly recall the definition of a  $\Gamma$ -space from [28] to fix some conventions:  $\Gamma$  is the category whose objects are all finite sets, and whose morphisms from  $S$  to  $T$  are the maps  $\theta : S \longrightarrow \mathcal{P}(T)$  such that  $\theta(\alpha)$  and  $\theta(\beta)$  are disjoint when  $\alpha \neq \beta$  ( $\mathcal{P}(T)$  denotes the set of all subsets of  $T$ ). The composite of  $\theta : S \longrightarrow \mathcal{P}(T)$  and  $\phi : T \longrightarrow \mathcal{P}(U)$  is given by

$$\psi : S \longrightarrow \mathcal{P}(U), \quad \psi(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \phi(\beta).$$

For each  $n \in \mathbb{Z}_{\geq 0}$ , we let  $\underline{n}$  denote the set  $\{1, \dots, n\}$ , where  $\underline{0}$  is defined to be the empty set. For nonnegative integers  $k \leq n$ , we let  $i_k : \underline{1} \longrightarrow \underline{n}$  be the morphisms in  $\Gamma$  given by  $i_k(1) = \{k\} \subset \underline{n}$ .

**Definition 5.3.** A  $\Gamma$ -space is a functor  $A : \Gamma^{\text{op}} \longrightarrow \text{Top}$ . It is called *special* if it satisfies the following conditions:

- (i)  $A(\underline{0})$  is weakly contractible;

- (ii) for any  $n \in \mathbb{Z}_{\geq 0}$ , the map  $p_n : A(\underline{n}) \longrightarrow A(\underline{1})^{\times n}$ , induced by the maps  $i_k : \underline{1} \longrightarrow \underline{n}$ , is a weak homotopy equivalence.

The space  $A(\underline{1})$  will often be referred to as the *underlying space* associated to the  $\Gamma$ -space  $A$ .

For a special  $\Gamma$ -space  $A$ , the set of path-components  $\pi_0(A(\underline{1}))$  obtains the structure of a monoid by via the operation defined by

$$\pi_0 A(\underline{1}) \times \pi_0 A(\underline{1}) = \pi_0(A(\underline{1}) \times A(\underline{1})) \xrightarrow{(p_2^*)^{-1}} \pi_0 A(\underline{2}) \xrightarrow{m_2} \pi_0 A(\underline{1}),$$

where  $m_2 : A(\underline{2}) \longrightarrow A(\underline{1})$  is the map induced by the morphism  $\underline{1} \longrightarrow \underline{2}$  taking  $\{1\}$  to  $\{1, 2\}$ . It follows easily that this monoid structure on  $\pi_0(A(\underline{1}))$  is always commutative.

**Definition 5.4.** A special  $\Gamma$ -space  $A$  is called *very special* if  $\pi_0(A(\underline{1}))$  is a group under the above composition.

The following theorem is obtained by assembling the results of [28].

**Theorem 5.5** (G. Segal '74). *Let  $A$  be a special  $\Gamma$ -space. If  $A$  is very special then the underlying space  $A(\underline{1})$  is an infinite loop space. In particular, the unit component of  $A(\underline{1})$  is an infinite loop space.*

We proceed to give  $|\mathbf{C}_\bullet^\mathcal{L}|$  the structure of a  $\Gamma$ -space. To this end we make every  $\mathbf{C}_p^\mathcal{L}$  into the underlying space of a  $\Gamma$ -space  $A_p$ . We roughly follow [21], where a similar construction is carried out for the ordinary cobordism category. In particular, it will be immediate that the forgetful map  $|\mathbf{C}_\bullet^\mathcal{L}| \rightarrow |\mathbf{C}_\bullet|$  refines to a map of  $\Gamma$ -spaces.

**Construction 5.1.** We define  $A_p(S)$  to be the subset of  $(\mathbf{C}_p^\mathcal{L})^S$  consisting of those tuples

$$(a^s, \varepsilon^s, W^s, \ell^s, L^s)_{s \in S}$$

that satisfy for all  $s, s' \in S$ :

- (a)  $a^s = a^{s'}$ , and
- (b) the submanifolds,  $W^s$  and  $W^{s'} \subset \mathbb{R} \times (-1, 1)^\infty$ , are disjoint.

For the rest of this proof, we will suppress  $\varepsilon$  and  $\ell$  from the notation and will denote elements of  $A_p(S)$  by tuples  $(a, W^s, L^s)$ . Given a morphism  $\phi : S \longrightarrow T$ , the map

$$A_p(\phi) : A_p(T) \longrightarrow A_p(S)$$

is defined by sending a tuple  $(a, W^t, L^t) \in A_p(T)$  to the element in  $A_p(S)$  whose entry in the  $s$ -th spot is:

$$(5.2) \quad \left( a, \bigsqcup_{t \in \phi(s)} W^t, \sum_{t \in \phi(s)} L^t \right).$$

It is readily checked that this new element indeed lies in  $A_p(S)$ . From these observations, it follows that  $A_p$  does indeed define a  $\Gamma$ -space.



In order to prove that it is special, we just need to check that for each  $\underline{k}$ , the map

$$p_k : A_p(\underline{k}) \longrightarrow A_p(\underline{1})^{\times k} = (\mathbf{C}_p^\mathcal{L})^{\times k}$$

is a weak homotopy equivalence. Notice that the map  $p_k$  is an embedding and is given by the formula

$$(a, (W^1, L^1), \dots, (W^k, L^k)) \mapsto ((a, (W^1, L^1)), \dots, (a, (W^k, L^k))).$$

Let  $A'_p(\underline{k})$  denote the subspace of  $(\mathbf{C}_p^\mathcal{L})^{\times k}$  consisting of those tuples,

$$(a^1, (W^1, L^1)), \dots, (a^k, (W^k, L^k)),$$

such that  $a^1 = \dots = a^k$  (but with no disjointness condition on the manifolds  $W^1, \dots, W^k$ ). The map  $p_k$  factors as the composition of inclusions

$$(5.3) \quad A_p(\underline{k}) \longrightarrow A'_p(\underline{k}) \longrightarrow A_p(\underline{1})^{\times k}.$$

The fact that the first inclusion is a weak homotopy equivalence follows from the *Whitney embedding theorem* together with the fact that the ambient space is infinite dimensional. The fact that the second inclusion is a weak homotopy equivalence follows from  $\mathbb{R}$  being contractible by scaling and translating the manifolds involved. We leave the details to the reader.

Now general facts about  $\Gamma$ -spaces due to Segal imply Theorem 4.13:

**Corollary 5.6.** *The geometric realization  $|\mathbf{C}_\bullet^\mathcal{L}|$  has the structure of a special  $\Gamma$ -space.*

*Proof.* For each  $p \in \mathbb{Z}_{\geq 0}$ , let  $\underline{k} \mapsto A_p(\underline{k})$  denote the  $\Gamma$ -space constructed in Lemma 5.1. The correspondence  $[p] \mapsto A_p(\underline{\phantom{x}})$  defines a semi-simplicial  $\Gamma$ -space (i.e. a contravariant functor from  $\Delta_{\text{inj}}$  to the category of  $\Gamma$ -spaces) which we denote by  $A_\bullet(\underline{\phantom{x}})$ . Taking level-wise geometric realization, we obtain the new  $\Gamma$ -space  $\underline{k} \mapsto |A_\bullet(\underline{k})|$ , and it is a general fact from [28] that simplicial, level-wise special  $\Gamma$ -spaces realise to special  $\Gamma$ -spaces. The corollary then follows from the fact that  $|A_\bullet(\underline{1})| = |\mathbf{C}_\bullet^\mathcal{L}|$ .  $\square$

## 6. ALTERNATE MODELS FOR THE NERVE

The category defined in Section 4.2 is difficult to analyze directly. In order to prove Theorems 4.10 - 4.12 we will need to work with a more flexible substitute for the semi-simplicial nerve.

**6.1. The main construction.** Recall the semi-simplicial space  $\mathbf{C}_\bullet$  from Definition 5.1 and  $\mathbf{C}_\bullet^\mathcal{L}$  from Definition 5.2. In the definition below we define a new semi-simplicial space  $\mathbf{D}_\bullet^\mathcal{L}$ .

**Definition 6.1.** For  $p \in \mathbb{Z}_{\geq 0}$ ,  $\mathbf{D}_p^\mathcal{L}$  is defined to be the space of tuples  $(a, \varepsilon, (W, \ell), V)$  subject to the following conditions:

- (i) The tuple  $(a, \varepsilon, (W, \ell))$  is an element of  $\mathbf{C}_p$  with the property that the pair  $(W|_{[a_i, a_{i+1}]}, W|_{a_{i+1}})$  is  $(n-1)$ -connected for all  $i = 0, \dots, p-1$ .

(ii)  $V = (V_0, \dots, V_p)$  is a  $p$ -tuple of subspaces

$$V_i \leq H_{n+1}^{\text{cpt}}(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}) \quad \text{for } i = 0, \dots, p,$$

subject to the following conditions:

- (a) For each  $i$ , the restriction  $V_i|_{a_i} \leq H_n(W|_{a_i})$  is a Lagrangian subspace (recall from Section 2.2 that  $V_i|_{a_i}$  is the image of  $V_i$  under the map  $j_i : H_{n+1}^{\text{cpt}}(W) \rightarrow H_n(W|_{a_i})$ ).
- (b) Let  $i \neq j$ . Then the subspace  $V_i|_{a_j} \leq H_n(W|_{a_j})$  is contained in the subspace  $V_j|_{a_j}$ .

To topologize  $\mathbf{D}_p^{\mathcal{L}}$  we use the following construction. Choose once and for all a family of increasing diffeomorphisms

$$(6.1) \quad \psi = \psi(a_0, \varepsilon_0, a_p, \varepsilon_p) : (0, 1) \xrightarrow{\cong} (a_0 - \varepsilon_0, a_p + \varepsilon_p),$$

varying smoothly in the data  $(a_0, \varepsilon_0, a_p, \varepsilon_p)$ . We then let

$$\bar{\psi} = \bar{\psi}(a_0, \varepsilon_0, a_p, \varepsilon_p) : (0, 1) \times \mathbb{R}^\infty \rightarrow (a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty$$

be the smooth family of diffeomorphisms given by the product,  $\psi \times \text{Id}_{\mathbb{R}^\infty}$ . For each  $i$ , we define a map  $\pi_i : \mathbf{D}_p^{\mathcal{L}} \rightarrow \Psi_\theta^\Delta((0, 1) \times \mathbb{R}^\infty)$  by

$$(a, \varepsilon, W, \ell, V) \mapsto (\bar{\psi}^{-1}(W|_{(a_0, \varepsilon_0, a_p, \varepsilon_p)}), \ell|_{(a_0, \varepsilon_0, a_p, \varepsilon_p)} \circ D\bar{\psi}, \bar{\psi}^{-1}(V_i)).$$

Using these maps we obtain an embedding  $\mathbf{D}_p^{\mathcal{L}} \hookrightarrow \mathbf{C}_p \times \Psi_\theta^\Delta((0, 1) \times \mathbb{R}^\infty)^{p+1}$  defined by the formula

$$(a, \varepsilon, W, \ell, V) \mapsto ((a, \varepsilon, W, \ell), \pi_0(a, \varepsilon, W, \ell, V), \dots, \pi_p(a, \varepsilon, W, \ell, V)).$$

By this embedding we topologize  $\mathbf{D}_p^{\mathcal{L}}$  as a subspace of  $\mathbf{C}_p \times \Psi_\theta^\Delta((0, 1) \times \mathbb{R}^\infty)^{p+1}$ .

**Remark 6.2.** Since any two families of diffeomorphisms (6.1) are isotopic the topology on  $\mathbf{D}_p^{\mathcal{L}}$  is independent of the choice of  $\psi$ .

For  $0 < i < p$ , the face maps  $d_i : \mathbf{D}_p^{\mathcal{L}} \rightarrow \mathbf{D}_{p-1}^{\mathcal{L}}$  are defined by

$$d_i(a, \varepsilon, W, \ell, V) = (a(i), \varepsilon(i), W, \ell, V(i))$$

where  $a(i)$ ,  $\varepsilon(i)$ , and  $V(i)$  are the  $(p-1)$ -tuples obtained by removing the  $i$ -th entry from the  $p$ -tuples  $a$ ,  $\varepsilon$ , and  $V$  respectively. For the face maps  $d_0, d_p : \mathbf{D}_p^{\mathcal{L}} \rightarrow \mathbf{D}_{p-1}^{\mathcal{L}}$ , a small change is needed in the definition. The map  $d_0$  is defined by

$$d_0(a, \varepsilon, W, \ell, V) = (a(0), \varepsilon(0), W, \ell, V(0)|_{(a_1 - \varepsilon_1, a_p + \varepsilon_p)}).$$

and  $d_p$  is defined by

$$d_p(a, \varepsilon, W, \ell, V) = (a(p), \varepsilon(p), W, \ell, V(p)|_{(a_0 - \varepsilon_0, a_{p-1} + \varepsilon_{p-1})}).$$

These face maps are continuous as a consequence of Proposition 2.9. With these face maps defined, the assignment  $[p] \mapsto \mathbf{D}_p^{\mathcal{L}}$  makes  $\mathbf{D}_\bullet^{\mathcal{L}}$  into a semi-simplicial space.

We filter  $\mathbf{D}_\bullet^\mathcal{L}$  by a sequence of sub-semi-simplicial spaces

$$(6.2) \quad \mathbf{D}_\bullet^{\mathcal{L},n} \subset \dots \subset \mathbf{D}_\bullet^{\mathcal{L},-1} \subset \mathbf{D}_\bullet^{\mathcal{L},D} \subset \mathbf{D}_\bullet^\mathcal{L},$$

defined analogously to (5.1).

We want to compare the semi-simplicial space  $\mathbf{D}_\bullet^\mathcal{L}$  with the semi-simplicial space  $\mathbf{C}_\bullet^\mathcal{L}$  defined in Definition 5.2. In order to map  $\mathbf{D}_\bullet^\mathcal{L}$  to  $\mathbf{C}_\bullet^\mathcal{L}$  we need the following proposition.

**Proposition 6.3.** *Let  $p \in \mathbb{Z}_{\geq 0}$ . For any  $(a, \varepsilon, (W, \ell), V) \in \mathbf{D}_p^\mathcal{L}$  the associated tuple*

$$(a, \varepsilon, (W, \ell), V_0|_{a_0}, \dots, V_p|_{a_p})$$

*is an element of  $\mathbf{C}_p^\mathcal{L}$ . Thus the correspondence*

$$(a, \varepsilon, (W, \ell), V) \mapsto (a, \varepsilon, (W, \ell), V_0|_{a_0}, \dots, V_p|_{a_p})$$

*yields a well defined semi-simplicial map  $\mathbf{D}_\bullet^\mathcal{L} \rightarrow \mathbf{C}_\bullet^\mathcal{L}$ .*

*Proof.* Let  $(a, \varepsilon, (W, \ell), V) \in \mathbf{D}_p^\mathcal{L}$  with  $V = (V_0, \dots, V_p)$ . We need to show that for all  $0 \leq i < p$

$$\iota^{\text{in}}(V_i|_{a_i}) = \iota^{\text{out}}(V_{i+1}|_{a_{i+1}}),$$

where

$$\iota^{\text{in}} : H_n(W|_{a_i}) \rightarrow H_n(W|_{[a_i, a_{i+1}]}) \quad \text{and} \quad \iota^{\text{out}} : H_n(W|_{a_{i+1}}) \rightarrow H_n(W|_{[a_i, a_{i+1}]})$$

are the maps induced by inclusion. Let  $x \in V_i|_{a_i}$ . Choose  $v \in V_i$  such that  $v|_{a_i} = x$ . By Definition 6.1 (condition (ii), part (b)), we have  $v|_{a_{i+1}} \in V_{i+1}|_{a_{i+1}}$ . Let

$$\bar{v} \in H_{n+1}(W|_{[a_i, a_{i+1}]}, W|_{a_i} \sqcup W|_{a_{i+1}})$$

denote the image of  $v$  under

$$H_{n+1}^{\text{cpt}}(W) \xrightarrow{j!} H_{n+1}(W|_{[a_i, a_{i+1}]}, W|_{a_i} \sqcup W|_{a_{i+1}})$$

where  $j : W|_{[a_i, a_{i+1}]} \rightarrow W$  is the inclusion, and  $j!$  is defined as in Section 2.2. By commutativity of the diagram

$$(6.3) \quad \begin{array}{ccccc} & & H_{n+1}^{\text{cpt}}(W) & & \\ & \swarrow j! & & \searrow \cdot|_{a_i+\nu} & \\ H_{n+1}(W|_{[a_i, a_{i+1}]}, W|_{a_i} \sqcup W|_{a_{i+1}}) & \xrightarrow{\partial} & H_n(W|_{a_i} \sqcup W|_{a_{i+1}}) & \xrightarrow{\text{pr}_\nu} & H_n(W|_{a_i+\nu}) \end{array}$$

for  $\nu = 0, 1$ , which one checks straight from the definition of the restriction maps, it follows that

$$(6.4) \quad \partial(\bar{v}) = v|_{a_i} + v|_{a_{i+1}},$$

where for  $\nu = 0, 1$ . By exactness of

$$H_{n+1}(W|_{[a_i, a_{i+1}]}, W|_{a_i} \sqcup W|_{a_{i+1}}) \xrightarrow{\partial} H_n(W|_{a_i} \sqcup W|_{a_{i+1}}) \xrightarrow{\iota^{\text{in}} + \iota^{\text{out}}} H_n(W|_{[a_i, a_{i+1}]}),$$

we then find

$$\iota^{\text{in}}(x) = \iota^{\text{in}}(v|_{a_i}) = -\iota^{\text{out}}(v|_{a_{i+1}}).$$

So we see that  $\iota^{\text{in}}(x) \in \iota^{\text{out}}(V_{i+1}|_{a_{i+1}})$  and thus we have proven

$$\iota^{\text{in}}(V_i|_{a_i}) \leq \iota^{\text{out}}(V_{i+1}|_{a_{i+1}}).$$

Exchanging indices shows that

$$\iota^{\text{in}}(V_i|_{a_i}) \geq \iota^{\text{out}}(V_{i+1}|_{a_{i+1}})$$

and thus  $\iota^{\text{in}}(V_i|_{a_i}) = \iota^{\text{out}}(V_{i+1}|_{a_{i+1}})$ . □

We let

$$(6.5) \quad \mathcal{F} : \mathbf{D}_{\bullet}^{\mathcal{L}} \longrightarrow \mathbf{C}_{\bullet}^{\mathcal{L}}, \quad (a, \varepsilon, W, \ell, V) \mapsto (a, \varepsilon, W, \ell, (V_0|_{a_0}, \dots, V_p|_{a_p})),$$

denote the semi-simplicial map defined as a result of Proposition 6.3. The following theorem is the main result of this section, its proof occupies Section 6.3.

**Theorem 6.4.** *The semi-simplicial map  $\mathbf{D}_{\bullet}^{\mathcal{L}} \longrightarrow \mathbf{C}_{\bullet}^{\mathcal{L}}$  induces the weak homotopy equivalences*

$$|\mathbf{D}_{\bullet}^{\mathcal{L}}| \simeq |\mathbf{C}_{\bullet}^{\mathcal{L}}|, \quad |\mathbf{D}_{\bullet}^{\mathcal{L}, D}| \simeq |\mathbf{C}_{\bullet}^{\mathcal{L}, D}|, \quad |\mathbf{D}_{\bullet}^{\mathcal{L}, l}| \simeq |\mathbf{C}_{\bullet}^{\mathcal{L}, l}|,$$

for all  $l \in \mathbb{Z}_{\geq 0}$ .

**6.2. Topological flag complexes.** The proof of Theorem 6.4 (given in the following subsection) will require the use of the simplicial technique introduced in [9, Section 6.2], which we now recall. In Section 10, we will in fact need a slight strengthening, but for now the current version will suffice.

**Definition 6.5.** Let  $X_{\bullet} \longrightarrow X_{-1}$  be an augmented semi-simplicial space. We say that it is an augmented topological flag complex if for each  $p \in \mathbb{Z}_{\geq 0}$ :

- (i) the map  $X_p \longrightarrow X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0$  to the  $(p+1)$ -fold fibred product is a homeomorphism onto its image, which is an open subset;
- (ii) a tuple  $(v_0, \dots, v_p) \in X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0$  is in  $X_p$  if and only if  $(v_i, v_j) \in X_1$  for all  $i < j$ .

If elements  $v, w \in X_0$  lie in the same fibre over  $X_{-1}$  and  $(v, w) \in X_1$ , we say that  $w$  is orthogonal to  $v$ . If  $X_{-1}$  is a single point, we omit the adjective augmented.

The following result is [9, Theorem 6.2].

**Theorem 6.6.** *Let  $X_{\bullet} \longrightarrow X_{-1}$  be an augmented topological flag complex. Suppose that the following are true:*

- (i) *The map  $\varepsilon : X_0 \longrightarrow X_{-1}$  has local lifts of any map from a disk.*
- (ii)  *$\varepsilon : X_0 \longrightarrow X_{-1}$  is surjective.*
- (iii) *For any  $x \in X_{-1}$  and any (non-empty) finite set  $\{v_1, \dots, v_m\} \subseteq \varepsilon^{-1}(x)$  there exists an element  $v \in \varepsilon^{-1}(x)$  such that  $(v_i, v) \in X_1$  for all  $i$ .*

*Then the map  $|X_{\bullet}| \longrightarrow X_{-1}$  is a weak homotopy equivalence.*

**6.3. Proof of Theorem 6.4.** We will only explicitly prove the weak homotopy equivalence  $|\mathbf{D}_\bullet^\mathcal{L}| \simeq |\mathbf{C}_\bullet^\mathcal{L}|$ . The other weak homotopy equivalences asserted in the theorem are established by repeating the exact same argument, which is largely formal. Our first step is to define an augmented bi-semi-simplicial space  $\mathbf{C}_{\bullet,\bullet}^\mathcal{L} \rightarrow \mathbf{C}_{\bullet,-1}^\mathcal{L}$ , with  $\mathbf{C}_{\bullet,-1}^\mathcal{L} = \mathbf{C}_\bullet^\mathcal{L}$ .

**Definition 6.7.** Let  $p \in \mathbb{Z}_{\geq 0}$  and let  $x = (a, \varepsilon, (W, \ell), L) \in \mathbf{C}_p^\mathcal{L}$ . For each  $q \in \mathbb{Z}_{\geq -1}$ , we define  $\mathbf{Z}_q(x)$  to be the set of tuples  $(V^0, \dots, V^q)$  subject to the following conditions:

- (i) Each  $(a, \varepsilon, (W, \ell), V^j)$  is an element of  $\mathbf{D}_p^\mathcal{L}$ . In other words for each  $j$ ,  $V^j = (V_0^j, \dots, V_p^j)$  is a  $(p+1)$ -tuple of subspaces of  $H_{n+1}^{\text{cpt}}(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})$ , subject to the conditions from Definition 6.1.
- (ii) The equality

$$V_i^j|_{a_i} = L_i$$

holds for all  $j = 0, \dots, q$  and  $i = 0, \dots, p$ . In other words,  $\mathcal{F}(a, \varepsilon, W, \ell, V^j) = (a, \varepsilon, W, \ell, L)$  for all  $j = 0, \dots, q$ , where recall that  $\mathcal{F}$  is the map from (6.5).

For  $p, q \in \mathbb{Z}_{\geq -1}$ , the space  $\mathbf{C}_{p,q}^\mathcal{L}$  is defined by  $\mathbf{C}_{p,q}^\mathcal{L} = \{(x, y) \mid x \in \mathbf{C}_p^\mathcal{L} \text{ and } y \in \mathbf{Z}_q(x)\}$ . The assignment  $[p, q] \mapsto \mathbf{C}_{p,q}^\mathcal{L}$  defines a bi-semi-simplicial space  $\mathbf{C}_{\bullet,\bullet}^\mathcal{L}$ . The forgetful maps

$$\mathbf{C}_{p,q}^\mathcal{L} \rightarrow \mathbf{C}_p^\mathcal{L}, \quad (x, y) \mapsto x,$$

define an augmented bi-semi-simplicial space  $\mathbf{C}_{\bullet,\bullet}^\mathcal{L} \rightarrow \mathbf{C}_{\bullet,-1}^\mathcal{L}$ , with  $\mathbf{C}_{\bullet,-1}^\mathcal{L} = \mathbf{C}_\bullet^\mathcal{L}$ .

**Remark 6.8.** By condition (i) in the above definition, it follows that  $\mathbf{Z}_q(x) \cong [\mathbf{Z}_0(x)]^{q+1}$  for all  $x$ . It follows automatically that the semi-simplicial set given by the correspondence  $[q] \mapsto \mathbf{Z}_q(x)$  is contractible whenever it is non-empty.

Notice that the semi-simplicial space  $\mathbf{C}_{\bullet,0}^\mathcal{L}$  is nothing but  $\mathbf{D}_\bullet^\mathcal{L}$ . Under this identification the forgetful map  $\mathbf{C}_{\bullet,0}^\mathcal{L} \rightarrow \mathbf{C}_\bullet^\mathcal{L}$  coincides with the map  $\mathcal{F} : \mathbf{D}_\bullet^\mathcal{L} \rightarrow \mathbf{C}_\bullet^\mathcal{L}$ . Inclusion of zero-simplices yields an embedding  $|\mathbf{D}_\bullet^\mathcal{L}| = |\mathbf{C}_{\bullet,0}^\mathcal{L}| \hookrightarrow |\mathbf{C}_{\bullet,\bullet}^\mathcal{L}|$ . To prove Theorem 6.4 it will suffice to prove that the maps

$$|\mathbf{C}_{\bullet,0}^\mathcal{L}| \hookrightarrow |\mathbf{C}_{\bullet,\bullet}^\mathcal{L}| \rightarrow |\mathbf{C}_{\bullet,-1}^\mathcal{L}|$$

are both weak homotopy equivalences, where the first map is given by inclusion of zero-simplices and the second is induced by the augmentation. We break this up into two steps, Lemma 6.9 and Lemma 6.10.

**Lemma 6.9.** *The map  $|\mathbf{C}_{\bullet,\bullet}^\mathcal{L}| \rightarrow |\mathbf{C}_{\bullet,-1}^\mathcal{L}|$  induced by the augmentation is a weak homotopy equivalence.*

*Proof.* We will apply Theorem 6.6 for each  $p \in \mathbb{Z}_{\geq 0}$  to show that the induced maps  $|\mathbf{C}_{p,\bullet}^\mathcal{L}| \rightarrow \mathbf{C}_{p,-1}^\mathcal{L}$  are weak homotopy equivalences for each  $p \in \mathbb{Z}_{\geq 0}$ . Geometrically realizing the first coordinate will then imply the lemma.

We will need to verify conditions (i), (ii), and (iii) from the statement of Theorem 6.6. Condition (i) is proven similarly to [9, Proposition 6.10] and so we omit the proof. Condition (iii) is essentially trivial (see Remark 6.8): Indeed, if  $x = (a, \varepsilon, W, \ell, L) \in \mathbf{C}_p^{\mathcal{L}}$  then by Definition 6.1 it follows that for any two elements  $V^1, V^2 \in \mathbf{Z}_0(x)$ , the pair  $(V^1, V^2)$  is an element of  $\mathbf{Z}_1(x)$ . Similarly if  $V^1, \dots, V^k \in \mathbf{Z}_0(x)$  is an arbitrary collection, if we set  $V = V^1$ , it follows that  $(V, V^i) \in \mathbf{Z}_1(x)$  for all  $i = 1, \dots, k$ . This establishes condition (iii).

We proceed to verify condition (ii). Let  $x = (a, \varepsilon, (W, \ell), L) \in \mathbf{C}_p^{\mathcal{L}}$  be with  $L = (L_0, \dots, L_p)$ . We will need to show that  $\mathbf{Z}_0(x)$  is non-empty. By the definition of  $\mathbf{C}_p^{\mathcal{L}}$ , we have

$$(6.6) \quad \iota^{\text{in}}(L_0) = \iota^{\text{out}}(L_1)$$

as subspaces of  $H_n(W|_{[a_0, a_1]})$ , where  $\iota^{\text{in}}$  and  $\iota^{\text{out}}$  are the maps induced by the inclusions

$$W|_{a_0} \hookrightarrow W|_{[a_0, a_1]} \hookleftarrow W|_{a_1}.$$

Let  $x_1, \dots, x_k \in L_0$  be a set of generators. By (6.6), for each  $i = 1, \dots, k$ , we may choose  $y_i \in L_1$  such that

$$\iota^{\text{in}}(x_i) = \iota^{\text{out}}(y_i).$$

By exactness of

$$H_{n+1}(W|_{[a_0, a_1]}, W|_{a_0} \sqcup W|_{a_1}) \xrightarrow{\partial} H_n(W|_{a_0} \sqcup W|_{a_1}) \xrightarrow{\iota^{\text{in}} + \iota^{\text{out}}} H_n(W|_{[a_0, a_1]}),$$

it follows that for each  $i = 1, \dots, k$ , there exists a class  $w_i \in H_{n+1}(W|_{[a_0, a_1]}, W|_{a_0} \sqcup W|_{a_1})$  such that

$$\partial w_i = x_i - y_i.$$

Since  $y_i \in L_1$  for all  $i$ , we can similarly find classes  $v_i \in H_{n+1}(W|_{[a_1, a_2]}, W|_{a_1} \sqcup W|_{a_2})$ , with  $\partial_{\text{in}}(v_i) = y_i$  and  $\partial_{\text{out}}(v_i) = -z_i$  for some classes  $z_i \in H_n(W|_{a_2})$  with  $\iota_{\text{in}}(y_i) = \iota_{\text{out}}(z_i)$ . Now consider the element  $j_0(w_i) + j_1(v_i) \in H_{n+1}(W|_{[a_0, a_2]}, W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2})$  where

$$j_\nu : H_{n+1}(W|_{[a_\nu, a_{\nu+1}]}, W|_{a_\nu} \sqcup W|_{a_{\nu+1}}) \longrightarrow H_{n+1}(W|_{[a_0, a_2]}, W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2})$$

is the obvious inclusion for  $\nu = 0, 1$ . In the long exact sequence

$$\begin{array}{ccc} \cdots & \longrightarrow & H_{n+1}(W|_{[a_0, a_2]}, W|_{a_0} \sqcup W|_{a_2}) \\ & & \downarrow \\ & & H_{n+1}(W|_{[a_0, a_2]}, W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2}) \\ & & \downarrow \partial \\ & & H_n(W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2}, W|_{a_0} \sqcup W|_{a_2}) \longrightarrow \cdots \end{array}$$

of the triple  $(W|_{a_0} \sqcup W|_{a_2}, W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2}, W|_{[a_0, a_2]})$  it is clearly mapped to zero:

$$\begin{aligned} \partial(j_0(w_i) + j_1(v_i)) &= \partial_{\text{in}}(w_i) + \partial_{\text{out}}(w_i) + \partial_{\text{in}}(v_i) + \partial_{\text{out}}(v_i) \\ &= c(x_i - y_i + y_i - z_i) \\ &= 0 \end{aligned}$$

where  $c : H_n(W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2}) \rightarrow H_n(W|_{a_0} \sqcup W|_{a_2}, W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2})$ , as  $c(x_i - z_i)$  comes from  $H_n(W|_{a_0} \sqcup W|_{a_2})$ . We can therefore pick preimages  $u_i \in H_{n+1}(W|_{[a_0, a_2]}, W|_{a_0} \sqcup W|_{a_2})$  of  $j_0(w_i) + j_1(v_i)$ . These satisfy  $\partial_{\text{out}} u_i = z_i$  and so we can repeat the process until we have constructed a subspace

$$V^0 \subseteq H_{n+1}(W|_{[a_0, a_p]}, W|_{a_0} \sqcup W|_{a_p}) \cong H_{n+1}^{\text{cpt}}(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})$$

By construction and commutativity of the diagram

$$\begin{array}{ccccc} & & H_{n+1}^{\text{cpt}}(W|_{(a_0 - \varepsilon_0, a_i + \varepsilon_i)}) & & \\ & \swarrow j_! & & \searrow \cdot|_{a_\nu} & \\ H_{n+1}(W|_{[a_0, a_i]}, W|_{a_0} \sqcup W|_{a_i}) & \xrightarrow{\partial} & H_n(W|_{a_0} \sqcup W|_{a_i}) & \xrightarrow{\text{pr}_\nu} & H_n(W|_{a_\nu}) \end{array}$$

for  $\nu = 0, i$  we find  $V^0|_{a_0} = L_0$  and  $V^0|_{a_i} \leq L_i$  for all  $i \neq 0$ .

The other required subspaces  $V^i$  are constructed in an entirely analogous fashion.  $\square$

With the above proposition established, the proof of Lemma 6.9 is complete.

**Lemma 6.10.** *The map  $|\mathbf{D}_\bullet^\mathcal{L}| = |\mathbf{C}_{\bullet,0}^\mathcal{L}| \hookrightarrow |\mathbf{C}_{\bullet,\bullet}^\mathcal{L}|$  induced by inclusion of zero-simplices is a weak homotopy equivalence.*

*Proof.* The proof of this lemma is similar to the argument from [9, Page 327]. In the final argument we will correct a small oversight of [9]. To begin, we define a modified version of the semi-simplicial space  $\mathbf{C}_\bullet^\mathcal{L}$  which we denote by  $\mathbf{C}'_\bullet$ .  $\mathbf{C}'_\bullet$  is defined in the same way as  $\mathbf{C}_\bullet^\mathcal{L}$  except that the usual inequalities  $a_i + \varepsilon_i < a_{i+1} - \varepsilon_{i+1}$ , are replaced by  $a_i \leq a_{i+1}$  so that the intervals  $[a_i - \varepsilon_i, a_i + \varepsilon_i]$  are allowed to overlap (in contrast to [9] we do still require  $\varepsilon_i > 0$  for all  $i = 0, \dots, p$ , however. This seems to conform with the argument they provide). The augmented bi-semi-simplicial space  $\mathbf{C}'_{\bullet,\bullet} \rightarrow \mathbf{C}'_{\bullet,-1}$  is defined similarly with  $\mathbf{C}'_{\bullet,-1} = \mathbf{C}'_\bullet$ . The inclusions  $\mathbf{C}'_{\bullet,\bullet} \hookrightarrow \mathbf{C}_{\bullet,\bullet}^\mathcal{L}$  and  $\mathbf{C}'_\bullet \hookrightarrow \mathbf{C}_\bullet^\mathcal{L}$  are readily seen to be weak homotopy equivalences, as is explained in [9]. To prove the lemma it is enough to show that  $|\mathbf{C}'_{\bullet,0}| \hookrightarrow |\mathbf{C}'_{\bullet,\bullet}|$  is a weak homotopy equivalence.

We will define a retraction  $r : |\mathbf{C}'_{\bullet,\bullet}| \rightarrow |\mathbf{C}'_{\bullet,0}|$  which is a weak homotopy equivalence. For each  $p, q \in \mathbb{Z}_{\geq 0}$  there is a map

$$(6.7) \quad h_{p,q} : \mathbf{C}'_{p,q} \rightarrow \mathbf{C}'_{(p+1)(q+1)-1,0}$$

given by considering  $p+1$  regular values, each equipped with  $(q+1)$  collections of subspaces of  $H_{n+1}^{\text{cpt}}(W)$ , as  $(p+1)(q+1)$  not-necessarily distinct regular values each equipped with a single



collection of subspaces of  $H_{n+1}^{\text{cpt}}(W)$ . For example, in the case that  $p = 1$  and  $q = 2$ , the map  $\mathbf{C}'_{1,2} \rightarrow \mathbf{C}'_{5,0}$  is given by sending

$$((a_0, a_1), (W, \ell), (L_0, L_1), (V_0^0, V_1^0), (V_0^1, V_1^1), (V_0^2, V_1^2))$$

to the element

$$((a_0, a_0, a_0, a_1, a_1, a_1), (W, \ell), (L_0, L_0, L_0, L_1, L_1, L_1), (V_0^0, V_0^1, V_0^2, V_1^0, V_1^1, V_1^2)),$$

where we have dropped the data  $\varepsilon = (\varepsilon_0, \varepsilon_1)$  from the notation to save space. Being able to do this is the very reason for having distinct subspaces  $V_i$  for every slice  $W|_{a_i}$ . There is also a map

$$(6.8) \quad \rho_{p,q} : \Delta^p \times \Delta^q \rightarrow \Delta^{(p+1)(q+1)-1} \subset \mathbb{R}^{(p+1)(q+1)}$$

with  $(i + (q + 1)j)$ th coordinate given by  $(t, s) \mapsto t_j s_i$ . Taking the product of these maps yields

$$(6.9) \quad r_{p,q} : \mathbf{C}'_{p,q} \times \Delta^p \times \Delta^q \rightarrow \mathbf{C}'_{(p+1)(q+1)-1,0} \times \Delta^{(p+1)(q+1)-1}$$

which glue together to give a map  $r : |\mathbf{C}'_{\bullet,\bullet}| \rightarrow |\mathbf{C}'_{\bullet,0}|$ . It is clear that this map is a retraction and thus the induced map on homotopy groups is surjective. Consider the augmentation map  $|\varepsilon| : |\mathbf{C}'_{\bullet,\bullet}| \rightarrow |\mathbf{C}'_{\bullet}|$  in the second bi-semi-simplicial coordinate. By Lemma 6.9, this map is a weak homotopy equivalence. The fact that  $r$  induces an injection on homotopy groups will follow once we prove that  $|\varepsilon| : |\mathbf{C}'_{\bullet,\bullet}| \rightarrow |\mathbf{C}'_{\bullet}|$  induces the same map on homotopy groups as

$$(6.10) \quad |\mathbf{C}'_{\bullet,\bullet}| \xrightarrow{r} |\mathbf{C}'_{\bullet,0}| \xrightarrow{|\varepsilon_0|} |\mathbf{C}'_{\bullet}|.$$

In [9] it is claimed that the two maps are in fact equal, but this is not true. It is, however, almost the case as we now explain. It turns out that the semi-simplicial structure on  $\mathbf{C}'_{\bullet}$  extends to that of a simplicial space: The  $i$ th degeneracy map  $s_i : \mathbf{C}'_p \rightarrow \mathbf{C}'_{p+1}$  is defined by sending the  $p$ -simplex

$$((a_0, \dots, a_i, \dots, a_p), (\varepsilon_0, \dots, \varepsilon_i, \dots, \varepsilon_p), W, \ell)$$

to the  $(p + 1)$ -simplex,  $((a_0, \dots, a_i, a_i, \dots, a_p), (\varepsilon_0, \dots, \varepsilon_i, \varepsilon_i, \dots, \varepsilon_p), W, \ell)$ . Let  $\|\mathbf{C}'_{\bullet}\|$  denote the geometric realization of the  $\mathbf{C}'_{\bullet}$  with this simplicial structure and let  $p : |\mathbf{C}'_{\bullet}| \rightarrow \|\mathbf{C}'_{\bullet}\|$  denote the quotient map defined by implementing the degeneracies into the geometric realization construction. It can be checked by hand that the diagram

$$(6.11) \quad \begin{array}{ccccc} |\mathbf{C}'_{\bullet,\bullet}| & \xrightarrow{r} & |\mathbf{C}'_{\bullet,0}| & \xrightarrow{|\varepsilon_0|} & |\mathbf{C}'_{\bullet}| \\ & \searrow |\varepsilon| & & & \downarrow p \\ & & |\mathbf{C}'_{\bullet}| & \xrightarrow{p} & \|\mathbf{C}'_{\bullet}\| \end{array}$$

is commutative. The injectivity of  $r_* : \pi_*(|\mathbf{C}'_{\bullet,\bullet}|) \rightarrow \pi_*(|\mathbf{C}'_{\bullet,0}|)$  will follow once it is established that the map  $p$  is a weak homotopy equivalence. According to [28, Proposition A.1], the fact that  $p$  is a weak homotopy equivalence follows from the fact that all degeneracy maps  $s_i : \mathbf{C}'_p \rightarrow \mathbf{C}'_{p+1}$  are cofibrations. To see that  $s_i$  is a (closed) cofibration it is enough (e.g. by combining [5, 3.13 and

3.26]) to show that  $s_i(\mathbf{C}'_p) \subset \mathbf{C}'_{p+1}$  is given as the vanishing set of some real-valued function (which is obvious) and has an open neighborhood  $U$ , that deformation retracts onto  $s_i(\mathbf{C}'_p)$  and is given as the non-vanishing set of some real valued function. Such a neighborhood of  $s_i(\mathbf{C}'_p)$  is given by the set

$$\{(a, \varepsilon, W, \ell) \in \mathbf{C}'_{p+1} \mid a_{i+1} - a_i < \min\{\varepsilon_i, \varepsilon_{i+1}\}\}.$$

The deformation retraction is given by moving  $a_{i+1}$  towards  $a_i$  and shrinking  $\varepsilon_{i+1}$  so that  $a_{i+1}^t - \varepsilon_{i+1}^t$  never exceeds  $a_i - \varepsilon_i$ . Commutativity of the diagram (6.11) then implies that  $r$  induces an injection on homotopy groups. Combining this with what was proven earlier implies that  $r$  is a weak homotopy equivalence.  $\square$

With the two lemmas above it follows that the maps  $|\mathbf{C}_{\bullet,0}^{\mathcal{L}}| \hookrightarrow |\mathbf{C}_{\bullet,\bullet}^{\mathcal{L}}| \longrightarrow |\mathbf{C}_{\bullet,-1}^{\mathcal{L}}|$  are weak homotopy equivalences. Identifying  $|\mathbf{D}_{\bullet}^{\mathcal{L}}| = |\mathbf{C}_{\bullet,0}^{\mathcal{L}}|$  and  $|\mathbf{C}_{\bullet,-1}^{\mathcal{L}}| = |\mathbf{C}_{\bullet}^{\mathcal{L}}|$ , establishes the weak homotopy equivalence  $|\mathbf{D}_{\bullet}^{\mathcal{L}}| \simeq |\mathbf{C}_{\bullet}^{\mathcal{L}}|$  and completes the proof of Theorem 6.4.

**6.4. Flexible models.** We will also need to work with an even more flexible model for the semi-simplicial spaces defined above. The following definition is similar to [9, Definition 2.8].

**Definition 6.11.** Define  $\mathbf{X}_{\bullet}^{\mathcal{L}}$  to be the semi-simplicial space with  $p$ -simplices consisting of certain tuples  $(a, \varepsilon, (W, \ell), (V_0, \dots, V_p))$  with  $a \in \mathbb{R}^{p+1}$ ,  $\varepsilon \in \mathbb{R}_{>0}^{p+1}$ , and

$$(W, \ell, V_i) \in \Psi_{\theta}^{\Delta}((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^{\infty}) \quad \text{for all } i = 0, \dots, p,$$

subject to the following conditions:

- (i)  $W$  is contained in  $(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times (-1, 1)^{\infty-1}$ ;
- (ii)  $a_{i-1} + \varepsilon_{i-1} < a_i - \varepsilon_i$  for  $i = 1, \dots, p$ ;
- (iii) for any two regular values  $b < c \in \cup_{i=0}^p (a_0 - \varepsilon_0, a_p + \varepsilon_p)$  of the height function  $W \longrightarrow \mathbb{R}$ , the pair  $(W|_{[b,c]}, W|_c)$  is  $(n-1)$ -connected.
- (iv) Let  $i = 0, \dots, p$ . If  $c \in (a_i - \varepsilon_i, a_i + \varepsilon_i)$  is a regular value for the height function  $W \longrightarrow \mathbb{R}$ , then  $V_i|_c \leq H_n(W|_c)$  is a Lagrangian subspace and  $V_j|_c \leq V_i|_c$  for all other  $j = 0, \dots, p$ .

The space  $\mathbf{X}_p^{\mathcal{L}}$  is topologized in a similar way as the space  $\mathbf{D}_p^{\mathcal{L}}$ . As in Definition 6.1 we fix once and for all a family of increasing diffeomorphisms

$$\psi = \psi(a_0, \varepsilon_0, a_p, \varepsilon_p) : (0, 1) \xrightarrow{\cong} (a_0 - \varepsilon_0, a_p + \varepsilon_p)$$

varying smoothly in the data  $(a_0, \varepsilon_0, a_p, \varepsilon_p)$ . Using this family of diffeomorphisms to reparametrise the interval  $(a_0 - \varepsilon_0, a_p + \varepsilon_p)$ , we may embed  $\mathbf{X}_p^{\mathcal{L}} \hookrightarrow \Psi_{\theta}^{\Delta}((0, 1) \times \mathbb{R}^{\infty})^{p+1}$ , and the topologize  $\mathbf{X}_p^{\mathcal{L}}$  as a subspace.

For  $0 < i < p$ , the face map  $d_i : \mathbf{X}_p^{\mathcal{L}} \longrightarrow \mathbf{X}_{p-1}^{\mathcal{L}}$  is defined by

$$d_i(a, \varepsilon, W, \ell, V) = (a(i), \varepsilon(i), W, \ell, V(i)).$$

The face map  $d_0 : \mathbf{X}_p^{\mathcal{L}} \rightarrow \mathbf{X}_{p-1}^{\mathcal{L}}$  is given by

$$d_0(a, \varepsilon, W, \ell, V) = (a(0), \varepsilon(0), W|_{(a_1 - \varepsilon_1, a_p + \varepsilon_p)}, \ell|_{(a_1 - \varepsilon_1, a_p + \varepsilon_p)}, V(0)|_{(a_1 - \varepsilon_1, a_p + \varepsilon_p)}),$$

$d_p : \mathbf{X}_p^{\mathcal{L}} \rightarrow \mathbf{X}_{p-1}^{\mathcal{L}}$  is defined similarly. Proposition 2.9 implies that these maps are continuous. The assignment  $[p] \mapsto \mathbf{X}_p^{\mathcal{L}}$  defines a semi-simplicial space  $\mathbf{X}_{\bullet}^{\mathcal{L}}$ .

Notice that the principle difference between  $\mathbf{X}_{\bullet}^{\mathcal{L}}$  and  $\mathbf{D}_{\bullet}^{\mathcal{L}}$  is that for any  $(a, \varepsilon, (W, \ell), V) \in \mathbf{X}_p^{\mathcal{L}}$  the manifold  $W$  is not required to be cylindrical over the intervals  $(a_i - \varepsilon_i, a_i + \varepsilon_i)$ . Furthermore these intervals need not even be comprised entirely of regular values for the height function. The formula

$$(6.12) \quad (a, \varepsilon, W, \ell, V) \mapsto (a, \varepsilon, W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, \ell|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, V)$$

induces a semi-simplicial map  $\mathbf{D}_{\bullet}^{\mathcal{L}} \rightarrow \mathbf{X}_{\bullet}^{\mathcal{L}}$  which is continuous by Proposition 2.9. As before we define a sequence of sub-semi-simplicial spaces of  $\mathbf{X}_{\bullet}^{\mathcal{L}}$  from (6.2).

**Definition 6.12.** We define a sequence of sub-semi-simplicial spaces of  $\mathbf{X}_{\bullet}^{\mathcal{L}}$  as follows.

- (a)  $\mathbf{X}_{\bullet}^{\mathcal{L}, D} \subset \mathbf{X}_{\bullet}^{\mathcal{L}}$  has as its  $p$ -simplices those  $(a, \varepsilon, (W, \ell), V)$  such that  $W$  contains

$$(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times D$$

and such that the restriction of  $\ell$  to  $(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times D$  agrees with the structure  $\ell_{\mathbb{R} \times D}$  used in the definition of  $\mathbf{Cob}_{\theta}^D$ .

- (b) Let  $l \in \mathbb{Z}_{\geq -1}$ .  $\mathbf{X}_{\bullet}^{\mathcal{L}, l} \subset \mathbf{X}_{\bullet}^{\mathcal{L}, D}$  has as its  $p$ -simplices those  $(a, \varepsilon, (W, \ell), V)$  with the property that for any regular value  $c \in \cup_{i=0}^p (a_i - \varepsilon_i, a_i + \varepsilon_i)$  of the height function, the manifold  $W|_c$  is  $l$ -connected.

The following proposition is proven by invoking [9, Proposition 2.20] and [8, Theorem 3.9]. We provide a sketch.

**Proposition 6.13.** *The map from (6.12) induces a weak homotopy equivalence,  $|\mathbf{D}_{\bullet}^{\mathcal{L}}| \simeq |\mathbf{X}_{\bullet}^{\mathcal{L}}|$ . Similarly, it induces weak equivalences  $|\mathbf{D}_{\bullet}^{\mathcal{L}, D}| \simeq |\mathbf{X}_{\bullet}^{\mathcal{L}, D}|$  and  $|\mathbf{D}_{\bullet}^{\mathcal{L}, l}| \simeq |\mathbf{X}_{\bullet}^{\mathcal{L}, l}|$ .*

*Proof sketch.* Let  $\mathbf{D}_{\bullet}^{\mathcal{L}, \natural}$  be the semi-simplicial space with  $p$ -simplices given by tuples  $(a, \varepsilon, (W, \ell), V)$  as in the definition of  $\mathbf{D}_{\bullet}^{\mathcal{L}}$ , but instead of requiring  $W$  to be cylindrical over the intervals  $(a_i - \varepsilon_i, a_i + \varepsilon_i)$ , we only require  $W$  to be *regular* over the intervals  $(a_i - \varepsilon_i, a_i + \varepsilon_i)$ . By this we mean that we require each interval  $(a_i - \varepsilon_i, a_i + \varepsilon_i)$  to consist entirely of regular values for the height function on  $W$ . There is an embedding  $\mathbf{D}_{\bullet}^{\mathcal{L}} \hookrightarrow \mathbf{D}_{\bullet}^{\mathcal{L}, \natural}$  and the map from the statement of the proposition factors as the composite,

$$|\mathbf{D}_{\bullet}^{\mathcal{L}}| \xrightarrow{(1)} |\mathbf{D}_{\bullet}^{\mathcal{L}, \natural}| \xrightarrow{(2)} |\mathbf{X}_{\bullet}^{\mathcal{L}}|.$$

The proposition follows from the fact that both maps in this composition are weak homotopy equivalences. The proof that map (1) is a weak homotopy equivalence proceeds exactly as [8, Theorem

3.9], while the proof that map (2) is a weak homotopy equivalence goes through in the same way as [9, Proposition 2.20].  $\square$

**6.5. Proof of Theorem 4.10.** In this section we prove Theorem 4.10 which asserts the weak homotopy equivalence  $BCob_\theta^{\mathcal{L},D} \simeq BCob_\theta^{\mathcal{L}}$ . By the results of the previous section it will suffice to prove that the inclusion of semi-simplicial spaces  $\mathbf{X}_\bullet^{\mathcal{L},D} \hookrightarrow \mathbf{X}_\bullet^{\mathcal{L}}$  induces a weak homotopy equivalence on geometric realization. This is proven in essentially the same way as [9, Proposition 2.16]. We again provide a sketch.

Let  $D \subset (-\frac{1}{2}, 0] \times (-1, 1)^{\infty-1}$  be the  $2n$ -dimensional disk from (3.1) used in the definitions of  $Cob_\theta^{\mathcal{L},D}$  and  $\mathbf{X}_\bullet^{\mathcal{L},D}$ . We let  $\bar{D} \subset [0, \frac{1}{2}) \times (-1, 1)^{\infty-1}$  denote the submanifold obtained by reflecting  $D$  across the hyperplane  $\{0\} \times (-1, 1)^{\infty-1}$ . Let  $\hat{D} \subset (-\frac{1}{2}, \frac{1}{2}) \times (-1, 1)^{\infty-1}$  denote the union  $D \cup \bar{D}$ . Since  $D$  was chosen so that it agrees with the submanifold  $(-1, 0] \times \partial D$  near  $\{0\} \times (-1, 1)^{\infty-1}$ , it follows that  $\hat{D}$  is a smooth submanifold diffeomorphic to  $S^{2n}$ . Choose a  $\theta$ -structure  $\ell_{\hat{D}}$  on  $\hat{D}$  that restricts to the chosen  $\theta$ -structure  $\ell_D$  on  $D$  from (3.1). Let  $\ell_{\mathbb{R} \times \hat{D}}$  denote the  $\theta$ -structure on  $\mathbb{R} \times \hat{D} \subset \mathbb{R} \times (-1, 1)^\infty$  induced from  $\ell_{\hat{D}}$ . Consider the diffeomorphism

$$\phi : \mathbb{R} \times (-1, 1) \times \mathbb{R}^{\infty-1} \xrightarrow{\cong} \mathbb{R} \times (\frac{1}{2}, 1) \times \mathbb{R}^{\infty-1}, \quad \phi(t, s, x) \mapsto (t, \frac{s+3}{4}, x).$$

Using  $\phi$  and  $\hat{D}$  we define for each  $p \in \mathbb{Z}_{\geq 0}$  a map

$$(6.13) \quad r_p : \mathbf{X}_p^{\mathcal{L}} \longrightarrow \mathbf{X}_p^{\mathcal{L},D}, \quad (a, \varepsilon, (W, \ell_W), V) \mapsto \left( a, \varepsilon, (\mathbb{R} \times \hat{D}) \cup \phi(W), \ell_{\mathbb{R} \times \hat{D}} \cup \phi_* \ell_W, \phi_*^W(V) \right),$$

where  $\phi^W : W \longrightarrow \mathbb{R} \times \hat{D} \cup \phi(W)$  is the embedding induced by  $\phi$  and

$$\phi_*^W(V) = (\phi_*^W(V_0), \dots, \phi_*^W(V_p)).$$

To see that this map is well defined, one simply needs to observe that the regular values  $c \in \cup_{i=0}^p (a_i - \varepsilon_i, a_i + \varepsilon_i)$  for the height functions  $W \longrightarrow \mathbb{R}$  and  $\mathbb{R} \times \hat{D} \cup \phi(W) \longrightarrow \mathbb{R}$  agree, and that  $\phi^W$  induces an isomorphism  $H_n(W|_c) \cong H_n(\{c\} \times \hat{D} \cup \phi(W)|_c)$ . Since  $\mathbb{R} \times \hat{D}$  is a cylinder, it follows that  $\mathbb{R} \times \hat{D} \cup \phi(W)$  satisfies the connectivity conditions from Definition 6.12 (condition (a)). It is clear that the maps  $r_p$  assemble to define a semi-simplicial map  $r : \mathbf{X}_\bullet^{\mathcal{L}} \longrightarrow \mathbf{X}_\bullet^{\mathcal{L},D}$ .

**Lemma 6.14.** *For each  $p \in \mathbb{Z}_{\geq 0}$ , the map  $r_p : \mathbf{X}_p^{\mathcal{L}} \longrightarrow \mathbf{X}_p^{\mathcal{L},D}$  is a homotopy inverse to the inclusion  $\mathbf{X}_p^{\mathcal{L},D} \longrightarrow \mathbf{X}_p^{\mathcal{L}}$ , and thus is a homotopy equivalence.*

*Proof sketch.* The homotopies  $r_p \circ i_p \simeq \text{Id}_{\mathbf{X}_p^{\mathcal{L},D}}$  and  $i_p \circ r_p \simeq \text{Id}_{\mathbf{X}_p^{\mathcal{L}}}$  are constructed in the exact same way as in the proof of [9, Proposition 2.16] (one can use the exact same formulas in our situation). One simply needs to check that the one-parameter families used (to build these required homotopies) yield valid elements of the spaces  $\mathbf{X}_p^{\mathcal{L},D}$  and  $\mathbf{X}_p^{\mathcal{L}}$ . This is simply a matter of checking their construction. We leave it to the reader.  $\square$

The above proposition implies that the inclusion  $\mathbf{X}_\bullet^{\mathcal{L},D} \hookrightarrow \mathbf{X}_\bullet^{\mathcal{L}}$  is a level-wise homotopy equivalence and thus induces the homotopy equivalence  $|\mathbf{X}_\bullet^{\mathcal{L},D}| \simeq |\mathbf{X}_\bullet^{\mathcal{L}}|$ . Combining with Proposition 6.13 and Theorem 6.4, this concludes the proof of Theorem 4.10.

## 7. SURGERY ON MANIFOLDS EQUIPPED WITH A LAGRANGIAN SUBSPACE

In this section we develop some technical results about surgery on manifolds that will be useful later on in Sections 8 and 9. Without loss of continuity, the reader could skip this section and come back to it when used in the proofs of the main theorems. For what follows, let  $M$  be a closed,  $2n$ -dimensional, oriented manifold and let

$$(7.1) \quad L \leq H_n(M)$$

be a Lagrangian subspace with respect to the intersection form  $(H_n(M), \lambda, \mu)$ . Let

$$(7.2) \quad \phi : S^k \times D^{2n-k} \longrightarrow M$$

be an embedding. Let  $\widetilde{M}$  denote the manifold obtained by performing surgery on  $M$  along the embedding  $\phi$  and let  $M'$  denote the complement  $M \setminus \phi(S^k \times \text{Int}(D^{2n-k}))$ . Let

$$H_n(M) \xleftarrow{\alpha} H_n(M') \xrightarrow{\beta} H_n(\widetilde{M})$$

denote the maps induced by inclusion. Consider the subspaces,

$$(7.3) \quad L' := \alpha^{-1}(L) \leq H_n(M') \quad \text{and} \quad \widetilde{L} := \beta(\alpha^{-1}(L)) \leq H_n(\widetilde{M}).$$

We will use this same notation throughout the rest of the section.

**Proposition 7.1.** *Let  $\phi : S^k \times D^{2n-k} \longrightarrow M$  be an embedding as in (7.2) and suppose that  $k \leq n-2$ . Then  $\widetilde{L} \leq H_n(\widetilde{M})$  is also a Lagrangian subspace with respect to  $(H_n(\widetilde{M}), \lambda, \mu)$ .*

*Proof.* By excision we have isomorphisms

$$(7.4) \quad \begin{aligned} H_*(M, M') &\cong H_*(S^k \times D^{2n-k}, S^k \times S^{2n-k-1}), \\ H_*(\widetilde{M}, M') &\cong H_*(D^{k+1} \times S^{2n-k-1}, S^k \times S^{2n-k-1}), \end{aligned}$$

and since  $k \leq n-2$  it follows that

$$0 = H_n(M, M') = H_{n+1}(M, M') = H_n(\widetilde{M}, M') = H_{n+1}(\widetilde{M}, M').$$

From the long exact sequence associated to the pairs  $(M, M')$  and  $(\widetilde{M}, M')$  it follows that the maps

$$H_n(M) \xleftarrow{\alpha} H_n(M') \xrightarrow{\beta} H_n(\widetilde{M})$$

are both isomorphisms. The maps  $\alpha$  and  $\beta$  are codimension-0 embeddings and thus they preserve both the intersection pairing  $\lambda$  and its refinement  $\mu$ . It follows that  $\widetilde{L} = \beta(\alpha^{-1}(L)) \leq H_n(\widetilde{M})$  is a Lagrangian subspace.  $\square$

We will need to prove an analogue of the above result in the case that  $k = n-1$ . This case is more delicate than the previous lemma because it is possible for an  $(n-1)$ -surgery on a  $2n$ -dimensional manifold to alter the homology groups in degree  $n$ . We will need to use a preliminary homological lemma that is lifted from [16, Lemma 5.6]. Let  $\phi : S^k \times D^{2n-k} \longrightarrow M$  be as in (7.2) and let  $x \in H_k(M)$  be the class determined by  $\phi|_{S^k \times \{0\}} : S^k \longrightarrow M$ .

**Lemma 7.2.** *Let  $j : H_{2n-k}(M) \longrightarrow H_{2n-k}(M, M')$  be the map induced by inclusion and let*

$$\alpha_{2n-k} \in H_{2n-k}(M, M') \cong \mathbb{Z}$$

*be the generator induced by the orientation on  $M$ . The map  $j$  is given by the formula*

$$j(y) = \lambda(x, y) \cdot \alpha_{2n-k}$$

*for all  $y \in H_{2n-k}(M)$ , where  $\lambda : H_k(M) \otimes H_{2n-k}(M) \longrightarrow \mathbb{Z}$  is the intersection pairing.*

The next corollary follows by considering the exact sequence associated to the pair  $(M, M')$ .

**Corollary 7.3.** *Let  $y \in H_{m-k}(M)$  be a class such that  $\lambda(x, y) = 0$ . Then the class  $y$  is in the image of the map  $H_{m-k}(M') \longrightarrow H_{m-k}(M)$  induced by inclusion.*

Below is the analogue of Proposition 7.1 for the case  $k = n - 1$ .

**Proposition 7.4.** *Set  $k = n - 1$ . Let  $\phi : S^k \times D^{2n-k} \longrightarrow M$  be as in (7.2). Then the subspace  $\tilde{L} \leq H_n(\tilde{M})$  is a Lagrangian subspace.*

*Proof.* Let  $x \in H_{n-1}(M)$  denote the class represented by the embedding  $\phi|_{S^{n-1} \times \{0\}} : S^{n-1} \longrightarrow M$ . Let  $x' \in H_{n-1}(M')$  be the unique class that maps to  $x$  under the map  $H_{n-1}(M') \longrightarrow H_{n-1}(M)$  induced by inclusion, which is easily seen to be an isomorphism by the long exact sequence associated to the pair  $(M, M')$ . We make a preliminary claim whose proof we postpone until after the proof of the current proposition.

**Claim 7.5.** *The subspace  $L' = \alpha^{-1}(L) \leq H_n(M')$  is Lagrangian.*

The proof of the proposition breaks down into two cases: the case where  $x$  is of infinite order and the case where  $x$  is of finite order. Claim 7.5 and the above observations will be used in the proof of both cases of the proposition.

**Case 1:** Suppose that the class  $x \in H_{n-1}(M)$  has infinite order. With Claim 7.5 established, to prove the proposition it will suffice to prove that the map  $\beta : H_n(M') \longrightarrow H_n(\tilde{M})$  is an isomorphism. Since  $x$  has infinite order it follows that  $x' \in H_{n-1}(M')$  has infinite order as well. Since the boundary map  $H_n(\tilde{M}, M') \rightarrow H_{n-1}(M')$  of the long exact sequence for the pair  $(\tilde{M}, M')$  sends a generator to  $x'$  it is injective. It follows that  $\beta : H_n(M') \longrightarrow H_n(\tilde{M})$  is surjective. Since  $H_{n+1}(\tilde{M}, M') = 0$ , it follows  $\beta$  is injective as well and thus an isomorphism.

**Case 2:** Suppose that  $x$  is of order  $m < \infty$ . It follows that the class  $x' \in H_{n-1}(M')$  (that maps to  $x$ ) has order  $m < \infty$  as well. As before  $x'$  generates the image of  $H_n(\tilde{M}, M') \rightarrow H_{n-1}(M')$  and using the same exact sequence as before we obtain

$$(7.5) \quad 0 \longrightarrow H_n(M') \xrightarrow{\beta} H_n(\tilde{M}) \longrightarrow \text{Ker}(\partial) \cong m \cdot \mathbb{Z} \longrightarrow 0$$

Let now  $\alpha_{n+1} \in H_{n+1}(M, M') \cong \mathbb{Z}$  denote the generator consistent with the orientation associated to  $M$  and let  $y' \in H_n(M')$  denote the class  $\partial(\alpha_{n+1})$ , where  $\partial : H_{n+1}(M, M') \longrightarrow H_n(M')$  is the

boundary map. (It is represented by  $f : \{0\} \times S^{n+1} \rightarrow M'$ .) We will need to use the following basic properties about the class  $y'$  whose proof we postpone until after the proof of the current proposition.

**Claim 7.6.** *The class  $y'$  has infinite order. Furthermore  $\lambda(y', v) = 0$  for all  $v \in H_n(M')$  and  $y' \in L'$ .*

Let  $\tilde{y} \in H_n(\widetilde{M})$  denote the image of  $y'$  under  $\beta : H_n(M') \rightarrow H_n(\widetilde{M})$ . Since  $\beta$  is injective it follows that  $\tilde{y}$  has infinite order. Furthermore it follows that  $\tilde{y} \in \tilde{L} = \beta(L')$  by Claim 7.6. We make one more observation about the class  $\tilde{y}$ . We claim that for any  $v \in H_n(\widetilde{M})$  the vanishing of  $\lambda(\tilde{y}, v)$  is equivalent to  $v \in \text{im}(\beta)$ : Indeed the map  $\lambda(\tilde{y}, \cdot) : H_n(\widetilde{M}) \rightarrow \mathbb{Z}$  annihilates the image of  $\beta$  by Claim 7.6 and therefore factors over  $\ker(\partial) \cong m \cdot \mathbb{Z}$  by exactness of (7.5). Also, it cannot be the null map, as the intersection pairing on  $H_n(\widetilde{M})$  is non-degenerate. As a non-zero homomorphism from one infinite cyclic group to another it is injective. These two facts imply that  $\lambda(\tilde{y}, v) = 0$  if and only if the image of  $v$  under  $H_n(\widetilde{M}) \rightarrow \ker(\partial) \subset H_n(\widetilde{M}, M')$  is equal to zero. This implies the claim.

We now are in a position to show that  $\tilde{L}$  is a Lagrangian subspace. Let  $w \in \tilde{L}^\perp$ . Since  $\tilde{y} \in \tilde{L}$ , we have  $\lambda(\tilde{y}, w) = 0$  and thus  $w = \beta(w')$  for some  $w' \in H_n(M')$ . Since  $\beta$  preserves the intersection pairing it follows that  $w' \in (L')^\perp$ . By Claim 7.5,  $L'$  is a Lagrangian subspace, so  $w' \in L'$  which yields  $w \in \tilde{L}$ .

This proves that  $\tilde{L}^\perp \leq \tilde{L}$ . Since  $\tilde{L}$  is by definition equal to  $\beta(L')$ ,  $\beta$  preserves the intersection pairing, and  $L'$  is an isotropic subspace (i.e.  $L' \leq (L')^\perp$ ), it follows that  $\tilde{L}$  is an isotropic subspace as well, so indeed  $\tilde{L}^\perp = \tilde{L}$ . The fact that  $\mu$  vanishes on  $\tilde{L}$  also follows by the selfintersection form being preserved by  $\beta$ .  $\square$

*Proof of Claim 7.5.* We prove that the subspace  $L' \leq H_n(M')$  is Lagrangian. This result will hold for both cases in the proof of the above proposition. Let us start with the inclusion  $(L')^\perp \leq L'$ . Let  $v \in (L')^\perp$  and let  $w \in L$ . By surjectivity of  $\alpha : H_n(M') \rightarrow H_n(M)$ , we choose  $w' \in L' = \alpha^{-1}(L)$  such that  $\alpha(w') = w$ . Since  $v \in (L')^\perp$  we have

$$0 = \lambda(v, w') = \lambda(\alpha(v), w).$$

Since  $w$  was arbitrary  $\alpha(v) \in L^\perp$  and since  $L$  is lagrangian it follows that  $\alpha(v) \in L$  and so  $v \in L' = \alpha^{-1}(L)$ . This proves  $(L')^\perp \leq L'$ .

For the other inclusion suppose that  $v, w \in L'$ . Since  $\alpha$  preserves the intersection pairing we have  $\lambda(v, w) = \lambda(\alpha(v), \alpha(w))$ . Since  $\alpha(v), \alpha(w) \in L$  and  $L$  is Lagrangian it follows that  $\lambda(v, w) = \lambda(\alpha(v), \alpha(w)) = 0$ . This proves that  $L'$  is isotropic. The same argument shows that  $\mu$  vanishes on  $L'$ .  $\square$

*Proof of Claim 7.6.* We begin by showing that the class  $y' = \partial(\alpha_{n+1}) \in H_n(M')$  has infinite order. By assumption, the class  $x \in H_{n-1}(M)$  has finite order. It follows that  $\lambda(x, v) = 0$  for all  $v \in$



$H_{n+1}(M)$ . It then follows from Lemma 7.2 that the map  $H_{n+1}(M) \longrightarrow H_{n+1}(M, M')$  is the zero map. By exactness the boundary map

$$\partial : H_{n+1}(M, M') \longrightarrow H_n(M')$$

is then injective. Since  $y' = \partial(\alpha_{n+1})$  (where  $\alpha_{n+1} \in H_{n+1}(M, M') \cong \mathbb{Z}$  is the generator) it follows that  $y'$  has infinite order.

Since  $y'$  is in the image of the boundary map  $\partial$ , it follows by exactness that  $y'$  is in the kernel of  $\alpha : H_n(M') \longrightarrow H_n(M)$ . It follows from this that  $y' \in \alpha^{-1}(L) = L'$ , since  $\alpha^{-1}(L)$  contains the kernel of  $\alpha$ . This establishes the third assertion of Claim 7.6. Let  $v \in H_n(M')$ . We have

$$\lambda(v, y') = \lambda(\alpha(v), \alpha(y')) = \lambda(\alpha(v'), 0) = 0.$$

This proves that  $\lambda(v, y') = 0$  for all  $v \in H_n(M')$ . □

## 8. SURGERY ON OBJECTS BELOW THE MIDDLE DIMENSION

Let  $l \in \mathbb{Z}_{\geq -1}$ . We proceed to prove Theorem 4.11 which asserts that there is a weak homotopy equivalence  $B\mathbf{Cob}_\theta^{\mathcal{L}, l-1} \simeq B\mathbf{Cob}_\theta^{\mathcal{L}, l}$  whenever  $l \leq n-1$  and the tangential structure  $\theta : B \longrightarrow BO(2n+1)$  is such that  $B$  is  $l$ -connected. By Theorem 6.4, it will suffice to prove the weak homotopy equivalence  $|\mathbf{D}_\bullet^{\mathcal{L}, l-1}| \simeq |\mathbf{D}_\bullet^{\mathcal{L}, l}|$ .

**8.1. A surgery move.** The proof of Theorem 4.11 will require us to use the surgery move introduced in [9, Section 4.2]. In this section we recall the basic properties of this surgery move. To begin, we denote  $d := 2n+1$  and fix a positive integer  $l < d$ . Choose a smooth function  $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$  which is the identity on  $(-\infty, \frac{1}{2})$ , has nowhere negative derivative, and  $\sigma(t) = 1$  for all  $t \geq 1$ . We define

$$(8.1) \quad K^l = \{(x, y) \in \mathbb{R}^{d-l} \times \mathbb{R}^{l+1} \mid |y|^2 = \sigma(|x|^2 - 1)\}.$$

Let  $h : K^l \longrightarrow \mathbb{R}$  denote the function defined by projecting onto the first coordinate in the ambient space. We will refer to this function as the *height function*. It is easy to see that this function has exactly two critical points, both of which are non-degenerate. The point  $(-1, 0, \dots, 0) \in K^l$  is a critical point of index  $l+1$  and  $(1, 0, \dots, 0) \in K^l$  is a critical point of index  $d-l-1$ . From the definition of  $\sigma$  we find that  $K^l$  is contained in  $\mathbb{R}^{d-l} \times D^{l+1}$ . Furthermore we have

$$K^l|_{(-6, -2)} = (-6, -2) \times \mathbb{R}^{d-l-1} \times S^l,$$

where as usual  $K^l|_{(-6, -2)}$  denotes the space  $h^{-1}(-6, -2)$ .

Let  $\ell$  be a  $\theta$ -structure on  $K^l$ . We define

$$(8.2) \quad (\mathcal{P}_t^l, \ell_t) \in \Psi_\theta((-6, -2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1}), \quad t \in [0, 1],$$

to be the family of manifolds constructed in [9, Section 4.2], with initial value given by

$$(\mathcal{P}_0^l, \ell_0) = (K^l|_{(-6, -2)}, \ell|_{(-6, -2)}).$$

By their construction, the correspondence  $\ell \mapsto (\mathcal{P}_t^l, \ell_t)$  defines a continuous map

$$\text{Bun}(TK^l, \theta^* \gamma) \longrightarrow \text{Maps} \left( [0, 1], \Psi_\theta((-6, -2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1}) \right).$$

The proposition below sums up the main properties of the one-parameter family (8.2) and is essentially a restatement of [9, Proposition 4.2]. This will be used extensively in Section 8.3.

**Proposition 8.1.** *Let  $l < d = 2n + 1$  and let  $\ell$  be a  $\theta$ -structure on  $K^l$ . The one-parameter family*

$$(\mathcal{P}_t^l, \ell_t) \in \Psi_\theta((-6, -2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1}), \quad t \in [0, 1],$$

*of (8.2) satisfies the following conditions:*

(i)  $(\mathcal{P}_0^l, \ell_0) = (K^l|_{(-6, -2)}, \ell|_{(-6, -2)})$ .

(ii) *For all  $t$ ,  $(\mathcal{P}_t^l, \ell_t)$  agrees with  $(K^l|_{(-6, -2)}, \ell|_{(-6, -2)})$  outside of the region*

$$(-5, -2) \times B_2^{d-l-1}(0) \times D^{l+1},$$

*where  $\ell|_{(-6, -2)}$  denotes the restriction of  $\ell$  to  $K^l|_{(-6, -2)}$ . We will denote*

$$\mathcal{P}^\partial := K^l|_{(-6, -2)} \setminus \left[ (-5, -2) \times B_2^{d-l-1}(0) \times D^{l+1} \right].$$

(iii) *For all  $t$  and each pair of regular values  $-6 < a < b < -2$  of the height function  $h : \mathcal{P}_t^l \longrightarrow \mathbb{R}$ , the pair  $(\mathcal{P}_t^l|_{[a, b]}, \mathcal{P}_t^l|_b)$  is  $(d - l - 2)$ -connected.*

(iv) *Let  $c$  be a regular value for the height function  $h : \mathcal{P}_t^l \longrightarrow (-6, -2)$ . The level set  $\mathcal{P}_t^l|_c$  takes one of the following two forms:*

(a) *there is a diffeomorphism  $\mathcal{P}_t^l|_c \cong \mathcal{P}_0^l|_c \text{ rel } \mathcal{P}^\partial$ , or*

(b)  *$\mathcal{P}_t^l|_c$  is obtained from  $\mathcal{P}_0^l|_c$  by surgery in degree  $l$ .*

(v) *The only critical value of  $h : \mathcal{P}_1^l \longrightarrow (-6, -2)$  is  $-4$ , and for  $c \in (-4, -2)$  the manifold  $\mathcal{P}_1^l|_c$  is obtained from  $\mathcal{P}_0^l|_c$  by surgery in degree  $l$  along the standard embedding*

$$D^{d-l-1} \times S^l \hookrightarrow \{c\} \times \mathbb{R}^{d-l-1} \times S^l = \mathcal{P}_0^l|_c.$$

**Remark 8.2.** It is easily verified from the construction in [9, Section 4] that the one-parameter family (8.2) is *locally generated by vector fields* in the sense of Definition 2.10. We will ultimately need to use this family to construct a one-parameter family of elements in  $\Psi_\theta^\Delta(\mathbb{R}^\infty)$ , see Construction 8.1. In view of Construction 2.2 and Proposition 2.12, this property (of being locally generated by vector fields) will be important for verifying the continuity of this one-parameter family.

**8.2. A semi-simplicial resolution.** We proceed to construct a semi-simplicial space similar to the one constructed in [9, Section 4.3]. First we must choose once and for all, smoothly in the data  $(a_i, \varepsilon_i, a_p, \varepsilon_p)$ , increasing diffeomorphisms

$$(8.3) \quad \psi = \psi(a_i, \varepsilon_i, a_p, \varepsilon_p) : (-6, -2) \xrightarrow{\cong} (a_i - \varepsilon_i, a_p + \varepsilon_p),$$

sending  $[-4, -3]$  linearly onto  $[a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i]$ . Recall from (8.1) the manifold

$$K^l \subset \mathbb{R}^{d-l} \times D^{l+1}.$$

**Definition 8.3.** Fix once and for all an infinite set  $\Omega$ . Let  $x = (a, \varepsilon, (W, \ell_W), V) \in \mathbf{D}_p^{\mathcal{L}, l-1}$ . With  $a = (a_0, \dots, a_p)$ , for each  $i$  we write  $M_i$  for the manifold  $W|_{a_i}$ . Define  $\mathbf{Y}_0^l(x)$  to consist of tuples  $(\Lambda, \delta, e, \ell)$ , where:

- $\Lambda \subset \Omega$  is a finite subset;
- $\delta : \Lambda \rightarrow [p]$  is a function;
- $e : \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1} \rightarrow \mathbb{R} \times (-1, 1)^{\infty-1}$  is an embedding;
- $\ell$  is a  $\theta$ -structure on  $\Lambda \times K^l$ .

For  $i \in [p]$ , we write  $\Lambda_i = \delta^{-1}(i)$  and

$$e_i : \Lambda_i \times (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \mathbb{R}^{d-l-1} \times D^{l+1} \rightarrow \mathbb{R} \times (-1, 1)^{\infty-1},$$

for the embedding obtained by restricting  $e$  and reparametrising using (8.3). We let  $\ell_i$  denote the restriction of  $\ell$  to  $\Lambda_i \times K|_{(-6, -2)}$ . This data is required to satisfy the following conditions:

- (i)  $e^{-1}(W) = \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times \partial D^{l+1}$ . We let

$$\partial e : \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times \partial D^{l+1} \rightarrow W$$

denote the embedding restricted to the boundary.

- (ii) Let  $i \in [p]$ . For  $c \in \bigcup_{k=i}^p (a_k - \varepsilon_k, a_k + \varepsilon_k)$ ,

$$(x_1 \circ e_i)^{-1}(c) = \Lambda_i \times \{c\} \times \mathbb{R}^{d-l-1} \times D^{l+1}.$$

- (iii) The restriction of  $\ell$  to  $\Lambda \times K|_{(-6, -2)}$  agrees with the composition

$$\ell_W \circ D(\partial e) : T(\Lambda \times K|_{(-6, -2)}) \rightarrow \theta^* \gamma.$$

For each  $i \in [p]$ , the restriction of  $\partial e$  to  $\Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times \partial D^{l+1}$  has its image in  $M_i = W|_{a_i}$  and thus yields the embedding

$$(8.4) \quad \partial e_i : \Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times \partial D^{l+1} \rightarrow M_i.$$

Let  $\widetilde{M}_i$  denote the manifold obtained by performing surgery on  $M_i$  along  $\partial e_i$ . We require:

- (iv) For all  $i \in [p]$ , the manifold  $\widetilde{M}_i$  is  $l$ -connected.

**Remark 8.4.** We emphasize that the list  $V = (V_0, \dots, V_p)$  associated to the element  $x \in \mathbf{D}_p^{\mathcal{L}, l-1}$  played no role in the definition of  $\mathbf{Y}_0^l(x)$ . In this way it follows that the above definition reduces to [9, Definition 4.3] upon forgetting  $V$ .

We now proceed to use the construction from Definition 8.3 to define an augmented bi-semi-simplicial space.

**Definition 8.5.** Let  $x = (a, \varepsilon, W, \ell_W, V) \in \mathbf{D}_p^{\mathcal{L}, l-1}$  and let  $q \in \mathbb{Z}_{\geq 0}$ . We define  $\mathbf{Y}_q^l(x) \subset \mathbf{Y}_0^l(x)^{q+1}$  to be the subset consisting of those  $(q+1)$ -tuples

$$((\Lambda^0, \delta^0, e^0, \ell^0), \dots, (\Lambda^q, \delta^q, e^q, \ell^q)) \in (\mathbf{Y}_0^l(x))^{\times(q+1)}$$

such that  $\text{Im}(e^i) \cap \text{Im}(e^j) = \emptyset$  whenever  $i \neq j$ , where  $\text{Im}(e^i)$  denotes the image of  $e^i$ . For each  $x$ , the assignment  $q \mapsto \mathbf{Y}_q^l(x)$  defines a semi-simplicial set  $\mathbf{Y}_\bullet^l(x)$ . We obtain a bi-semi-simplicial set  $\mathbf{D}_{\bullet, \bullet}^{\mathcal{L}, l}$  by setting

$$\mathbf{D}_{p,q}^{\mathcal{L}, l} = \{(x, y) \mid x \in \mathbf{D}_p^{\mathcal{L}, l-1}, y \in \mathbf{Y}_q^l(x)\}.$$

We topologize  $\mathbf{D}_{p,q}^{\mathcal{L}, l}$  as a subspace of

$$\mathbf{D}_p^{\mathcal{L}, l-1} \times \left( \prod_{\Lambda \subset \Omega} C^\infty \left( \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1}, \mathbb{R} \times (-1, 1)^{\infty-1} \right) \right)^{(p+1)(q+1)}$$

and thus the assignment  $[p] \times [q] \mapsto \mathbf{D}_{p,q}^{\mathcal{L}, l}$  defines a bi-semi-simplicial space.

For each  $q$  we have a forgetful projection  $\mathbf{D}_{p,q}^{\mathcal{L}, l} \longrightarrow \mathbf{D}_p^{\mathcal{L}, l-1}$  yielding an augmented bi-semi-simplicial space  $\mathbf{D}_{\bullet, \bullet}^{\mathcal{L}, l} \longrightarrow \mathbf{D}_{\bullet, -1}^{\mathcal{L}, l}$ , with  $\mathbf{D}_{\bullet, -1}^{\mathcal{L}, l} = \mathbf{D}_\bullet^{\mathcal{L}, l-1}$ . Since the definition of  $\mathbf{D}_{\bullet, \bullet}^{\mathcal{L}, l}$  did not make use of the data  $V = (V_0, \dots, V_p)$  associated to an element of  $\mathbf{D}_\bullet^{\mathcal{L}, l}$ , a proof of the following result is obtained by repeating the steps of the proof of [9, Theorem 4.5] verbatim. We therefore omit its proof.

**Theorem 8.6.** *Let  $l \leq n-1$  and suppose that  $\theta : B \longrightarrow BO(2n+1)$  is such that  $B$  is  $l$ -connected. Then there are weak homotopy equivalences*

$$|\mathbf{D}_{\bullet, 0}^{\mathcal{L}, l}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet, \bullet}^{\mathcal{L}, l}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet, -1}^{\mathcal{L}, l}| = |\mathbf{D}_\bullet^{\mathcal{L}, l-1}|,$$

where the first map is induced by inclusion of zero-simplices and the second is induced by the augmentation.

**8.3. Implementation of the surgery move.** We now show how to implement the surgery move that was constructed in Section 8.1. Recall the map

$$\text{Bun}(TK^l, \theta^* \gamma^{2n+1}) \longrightarrow \text{Maps}([0, 1], \Psi_\theta((-6, -2) \times \mathbb{R}^{2n+1})), \quad \ell \mapsto (\mathcal{P}_t^l, \ell_t).$$

We use this to define a semi-simplicial map  $\mathbf{D}_{\bullet, 0}^{\mathcal{L}, l} \longrightarrow \mathbf{X}_\bullet^{\mathcal{L}, l-1}$ .

**Construction 8.1.** For what follows let

$$x = (a, \varepsilon, (W, \ell_W), V) \in \mathbf{D}_p^{\mathcal{L}, l-1} \quad \text{and} \quad y = (\Lambda, \delta, e, \ell) \in \mathbf{Y}_0^l(x).$$

In this way the pair  $(x, y)$  is an element of the space  $\mathbf{D}_{p,0}^{\mathcal{L},l}$ . For each  $i = 0, \dots, p$ , we have an embedding  $e_i$  and a  $\theta$ -structure  $\ell_i$  on  $\Lambda_i \times K^l|_{(-6,-2)}$ , where  $\Lambda_i = \delta^{-1}(i)$ . From this data we may construct a one-parameter family of  $\theta$ -manifolds

$$(8.5) \quad \mathcal{K}_{e_i, \ell_i}^t(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty)$$

as follows. First, using the reparametrisation (8.3) we may reparametrise the first coordinate of the family  $(\mathcal{P}_t^l, \ell_i) \in \Psi_\theta((-6, -2) \times \mathbb{R}^{2n+1})$  to obtain a new family

$$(\bar{\mathcal{P}}_t^l, \bar{\ell}_i) \in \Psi_\theta((a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{2n+1}).$$

This one-parameter family has all of the same properties as in Proposition 8.1, except now properties (iv) and (v) apply to the interval  $(a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$  (recall that the reparametrisation (8.3) sends  $(-4, -3)$  to  $(a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ ). We define  $\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W)$  to be equal to  $W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}$  outside of the image of  $e_i$ , on

$$e_i(\Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l} \times D^{l+1})$$

it is given by  $e_i(\Lambda_i \times \bar{\mathcal{P}}_t^l)$ .

We need to describe how to transport the subspaces  $V_0, \dots, V_p \leq H_{n+1}^{\text{cpt}}(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})$  over to the homology group  $H_{n+1}^{\text{cpt}}(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W))$ . Let  $W'$  denote the complement  $W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)} \setminus \text{Im}(e_i)$ . For each  $t \in [0, 1]$ , let

$$H_{n+1}^{\text{cpt}}(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}) \xleftarrow{\alpha} H_{n+1}^{\text{cpt}}(W') \xrightarrow{\beta^t} H_{n+1}^{\text{cpt}}(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W))$$

denote the maps induced by inclusion. The inclusions of  $W'$  are both proper maps and so the homomorphisms  $\alpha$  and  $\beta_t$  are indeed well-defined.

For  $t \in [0, 1]$  and  $j = \{0, \dots, p\}$  let

$$(8.6) \quad V_j^t \leq H_{n+1}^{\text{cpt}}(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W))$$

be the subspace given by

$$\beta^t(\alpha^{-1}(V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})) \leq H_{n+1}^{\text{cpt}}(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W)).$$

We have the following proposition.

**Proposition 8.7.** *The above construction defines a (continuous) one-parameter family*

$$(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W), V_j^t) \in \Psi_\theta^\Delta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty), \quad t \in [0, 1],$$

with initial value given by

$$(\mathcal{K}_{e_i, \ell_i}^0(W, \ell_W), V_j^0) = (W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, \ell|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}).$$

*Proof.* The condition

$$\mathcal{K}_{e_i, \ell_i}^0(W, \ell_W) = (W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, \ell|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})$$

follows immediately from the above construction. The equality

$$V_j^0 = \beta^0(\alpha^{-1}(V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}))$$

holds by definition of  $V_j^0$ . The maps  $\beta^0$  and  $\alpha$  agree and so  $V_j^0 = \alpha(\alpha^{-1}(V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}))$ . To prove that

$$V_j^0 = \alpha(\alpha^{-1}(V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})) = V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)},$$

it will suffice to show that  $\alpha : H_{n+1}^{\text{cpt}}(W') \rightarrow H_{n+1}^{\text{cpt}}(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})$  maps surjectively onto the subspace  $V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}$  (or in other words, every element of this subspace is in the image of  $\alpha$ ). This fact is established by a calculation in the proof of Lemma 8.10 (we actually prove that  $\alpha$  is surjective, see (8.12)). Continuity of the family  $(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W), V_j^t)$  follows from Proposition 2.12, using the fact that the family (8.2) is locally generated by vector fields (see Remark 8.2).  $\square$

In the results below (Propositions 8.8 - 8.12), we make some basic observations about the above construction, in particular we will verify that each  $V_j^t|_{a_j} \leq H_n(W|_{a_j})$  is indeed a langrangian. We must fix some notation first. For what follows, let

$$((a, \varepsilon, W, \ell_W, V), e, \ell) \in \mathbf{D}_{p,0}^{\mathcal{L}, n-1}.$$

Fix  $i \in \{0, \dots, p\}$  and let  $(W_t^i, \ell_t^i)$  denote the family of  $\theta$ -manifolds  $\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W)$  defined in (8.5). For each  $j = 0, \dots, p$ , let

$$V_j^t \leq H_{n+1}^{\text{cpt}}(W_t^i)$$

denote the subspace defined in (8.6). We denote by  $h : W_t^i \rightarrow \mathbb{R}$  the *height function* on  $W_t^i$  given by projecting  $W_t^i \subset (a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^{\infty-1}$  onto the first coordinate of the ambient space. All of our constructions will take place in  $(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty$  and so we will abuse notation and denote

$$(8.7) \quad W := W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}$$

(there won't be any conflict in the following propositions where this notation is used). Propositions 8.8 - 8.12 stated below will all use this same one-parameter family  $(W_t^i, \ell_t^i)$ . The integer  $i$  will remain fixed throughout.

**Proposition 8.8.** *Let  $c \in \cup_{j=0}^p (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j)$  be a regular value for the height function  $h : W_t^i \rightarrow \mathbb{R}$ . Then  $c$  is also a regular value for  $h : W \rightarrow \mathbb{R}$  and one of the following two scenarios applies to the level set  $W_t^i|_c$ :*

- (a) *there is a diffeomorphism  $W_t^i|_c \cong W|_c$ , rel  $W'|_c$ , or*
- (b)  *$W_t^i|_c$  is obtained from  $W|_c$  by a collection of surgeries of degree  $l$ .*

*Proof.* This follows straight from the definition of the surgery moves and property (iv) of 8.1.  $\square$

The next proposition records what happens to the regular level sets  $W_t^i|_c$ , specifically for  $c \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$  and when  $t = 1$ .

**Proposition 8.9.** *Let  $t = 1$  and let  $c \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$  be a regular value for the height function. Then  $W_1^i|_c$  is  $l$ -connected and is obtained from  $W|_c$  by a collection of surgeries in degree  $l$ .*

*Proof.* This again follows straight from property (iv) of 8.1.  $\square$

We now proceed to describe the behavior of the subspaces  $V_j^t \leq H_{n+1}^{\text{cpt}}(W_t^i)$ . We will need the following lemma.

**Lemma 8.10.** *Let  $c \in \cup_{k=0}^p (a_k - \varepsilon_k, a_k + \varepsilon_k)$  be a regular value for the height function  $h : W_t^i \rightarrow \mathbb{R}$ . Let*

$$H_n(W|_c) \xleftarrow{\alpha_c} H_n(W'|_c) \xrightarrow{\beta_c^t} H_n(W_t^i|_c)$$

*denote the maps induced by inclusion. Then for any  $j = 0, \dots, p$ , the two subspaces*

$$V_j^t|_c \leq H_n(W_t^i|_c) \quad \text{and} \quad \beta_c^t(\alpha_c^{-1}(V_j|_c)) \leq H_n(W_t^i|_c)$$

*are equal.*

*Proof.* Let

$$\pi_c : H_{n+1}^{\text{cpt}}(W) \rightarrow H_n(W|_c) \quad \text{and} \quad \pi'_c : H_{n+1}^{\text{cpt}}(W') \rightarrow H_n(W'|_c)$$

denote the restriction maps. To prove the lemma it will suffice to show that

$$\alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j)).$$

With this equality established, the proof of the proposition follows from commutativity of the diagram,

$$\begin{array}{ccc} H_{n+1}^{\text{cpt}}(W') & \xrightarrow{\beta^t} & H_{n+1}^{\text{cpt}}(W_t^i) \\ \downarrow \pi'_c & & \downarrow \\ H_n(W'|_c) & \xrightarrow{\beta_c^t} & H_n(W_t^i|_c). \end{array}$$

To show the equality  $\alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j))$  we need to make some calculations. Recall from (8.7) the notation

$$W = W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}.$$

We will need to use the exact sequence on  $H_*^{\text{cpt}}$  associated to the pair  $(W, W')$ . We never explicitly proved existence of such exact sequences on  $H_*^{\text{cpt}}$  and so we must justify its existence here. Since  $W$  is cylindrical outside of the subset  $W|_{[a_0, a_p]}$ , it follows that there are isomorphisms

$$\begin{aligned} (8.8) \quad H_k^{\text{cpt}}(W') &\cong H_k(W'|_{[a_0, a_p]}, W'|_{a_0} \cup W'|_{a_p}) \\ H_k^{\text{cpt}}(W) &\cong H_k(W|_{[a_0, a_p]}, W|_{a_0} \cup W|_{a_p}), \\ H_k^{\text{cpt}}(W, W') &\cong H_k(W|_{[a_0, a_p]}, W'|_{[a_0, a_p]} \cup W|_{a_0} \cup W|_{a_p}), \end{aligned}$$

for all degrees  $k$ . Combined with (8.8), the exact sequence in homology associated to the triad

$$(W|_{[a_0, a_p]}; W'|_{[a_0, a_p]}, W|_{a_0} \sqcup W|_{a_p})$$

yields the exact sequence

$$\cdots \longrightarrow H_{n+2}^{\text{cpt}}(W, W') \xrightarrow{\partial} H_{n+1}^{\text{cpt}}(W') \xrightarrow{\alpha} H_{n+1}^{\text{cpt}}(W) \longrightarrow H_{n+1}^{\text{cpt}}(W, W') \longrightarrow \cdots$$

Let  $P_i$  and  $P_i^\partial$  denote the manifolds

$$(8.9) \quad \begin{aligned} P_i &= \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times S^l \times \mathbb{R}^{2n-l}, \\ P_i^\partial &= \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times S^l \times (\mathbb{R}^{2n-l} \setminus D^{2n-l}). \end{aligned}$$

Since  $l \leq n-1$ , it follows that

$$(8.10) \quad H_k^{\text{cpt}}(P_i, P_i^\partial) = 0 \quad \text{for all } k < 2n-l+1.$$

By excision we have

$$(8.11) \quad H_k^{\text{cpt}}(W, W') \cong H_k^{\text{cpt}}(P_i, P_i^\partial) \quad \text{for all } k.$$

Again, we never explicitly proved the existence of the excision for  $H_*^{\text{cpt}}$  in the Section 2.2. Its usage here is justified again by the fact that  $W$  and  $P$  both have cylindrical ends and that the homology group  $H_k^{\text{cpt}}(W, W')$  can be identified with a relative homology group (without compact supports) via (8.8).

Combining (8.11) with (8.10), the above exact sequence implies that the map

$$(8.12) \quad \alpha : H_{n+1}^{\text{cpt}}(W') \longrightarrow H_{n+1}^{\text{cpt}}(W)$$

is an isomorphism in the case that  $l < n-1$ , and is surjective in the case that  $l = n-1$ . With this established, the verification of the equality  $\alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j))$  breaks down into two cases: the case where  $l < n-1$  and the case that  $l = n-1$ .

**Case 1:** Let  $l < n-1$ . We desire to show that  $\alpha_c : H_n(W'|_c) \longrightarrow H_n(W|_c)$  is an isomorphism. With this, the equality  $\alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j))$  will follow from the commutative diagram

$$\begin{array}{ccc} H_{n+1}^{\text{cpt}}(W) & \xrightarrow[\cong]{\alpha^{-1}} & H_{n+1}^{\text{cpt}}(W') \\ \downarrow \pi_c & & \downarrow \pi'_c \\ H_n(W|_c) & \xrightarrow[\cong]{\alpha_c^{-1}} & H_n(W'|_c). \end{array}$$

To prove that  $\alpha_c$  is an isomorphism, we need to analyze the pair  $(W|_c, W'|_c)$ . This pair takes on two forms depending on whether or not  $c$  is contained in the interval  $(a_i - \varepsilon_i, a_p + \varepsilon_p)$ . Let us first suppose that  $c \in (a_i - \varepsilon_i, a_p + \varepsilon_p)$ . In this case we have

$$(8.13) \quad \begin{aligned} P_i|_c &= \Lambda_i \times \{c\} \times S^l \times \mathbb{R}^{2n-l}, \\ P_i^\partial|_c &= \Lambda_i \times \{c\} \times S^l \times (\mathbb{R}^{2n-l} \setminus D^{2n-l}), \end{aligned}$$



where  $P$  and  $P^\partial$  are from (9.9). Since  $l < n - 1$ , it follows that  $H_k(P_i|_c, P_i^\partial|_c) = 0$  for all  $k \leq n + 1$ . Excision for the pair  $(W|_c, W'|_c)$  yields

$$H_k(W|_c, W'|_c) \cong H_k(P|_c, P^\partial|_c) \quad \text{for all } k,$$

and thus we obtain

$$H_{n+1}(W|_c, W'|_c) \quad \text{for all } k \leq n + 1.$$

From the exact sequence associated to  $(W|_c, W'|_c)$  it follows that

$$\alpha_c : H_n(W'|_c) \xrightarrow{\cong} H_n(W|_c)$$

is an isomorphism whenever  $c \in (a_i - \varepsilon_i, a_p + \varepsilon_p)$  (assuming  $l < n - 1$ ).

Now suppose that  $c \notin (a_i - \varepsilon_i, a_p + \varepsilon_p)$ . In this case  $W|_c = W'|_c$ , and so  $H_k(W|_c, W'|_c) = 0$  for all  $k$ . It follows again by the exact sequence associated to  $(W|_c, W'|_c)$  that

$$\alpha_c : H_n(W'|_c) \xrightarrow{\cong} H_n(W|_c)$$

is an isomorphism. This concludes the proof of the lemma in case 1, when  $l < n - 1$ .

**Case 2:** Suppose that  $l = n - 1$ . In this case the maps  $\alpha$  and  $\alpha_c$  are not necessarily isomorphisms and so we cannot employ the same argument used above. Consider the commutative diagram

$$(8.14) \quad \begin{array}{ccccccc} 0 & \longleftarrow & H_{n+1}^{\text{cpt}}(W) & \xleftarrow{\alpha} & H_{n+1}^{\text{cpt}}(W') & \xleftarrow{\partial} & H_{n+2}^{\text{cpt}}(W, W') \\ & & \pi_c \downarrow & & \downarrow \pi'_c & & \downarrow \bar{\pi}_c \\ 0 & \longleftarrow & H_n(W|_c) & \xleftarrow{\alpha_c} & H_n(W'|_c) & \xleftarrow{\partial_c} & H_{n+1}(W|_c, W'|_c) \end{array}$$

which has exact rows. To establish  $\alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j))$ , it will suffice to prove that the right-vertical map  $\bar{\pi}_c$  is surjective. Indeed, with surjectivity of  $\bar{\pi}_c$  established, the equality  $\alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j))$  can be verified through a simple diagram chase. The map  $\bar{\pi}_c$  takes on two forms depending on whether or not  $c$  is contained in the interval  $(a_i - \varepsilon_i, a_p + \varepsilon_p)$ .

Suppose that  $c \in (a_i - \varepsilon_i, a_p + \varepsilon_p)$ . By (9.9) and (8.13) it follows that the restriction map

$$H_{n+2}^{\text{cpt}}(P_i, P_i^\partial) \longrightarrow H_{n+1}^{\text{cpt}}(P_i|_c, P_i^\partial|_c)$$

is an isomorphism. By the commutativity of the diagram

$$\begin{array}{ccc} H_{n+2}^{\text{cpt}}(W, W') & \xrightarrow{\cong} & H_{n+2}(P_i, P_i^\partial) \\ \downarrow \bar{\pi}_c & & \downarrow \cong \\ H_{n+1}(W|_c, W'|_c) & \xrightarrow{\cong} & H_{n+1}(P_i|_c, P_i^\partial|_c) \end{array}$$

it follows that  $\bar{\pi}_c$  is an isomorphism, and hence surjective. This establishes the first case. Suppose now that  $c \notin (a_i - \varepsilon_i, a_p + \varepsilon_p)$ . Since  $\text{Im}(e_i) \cap W$  is contained in  $W|_{(a_i - \varepsilon_i, a_p + \varepsilon_p)}$ , it follows that  $W'|_c = W|_c$  and thus

$$H_k(W|_c, W'|_c) = 0 \quad \text{for all } k, \text{ when } c \notin (a_i - \varepsilon_i, a_p + \varepsilon_p).$$

It follows that  $\bar{\pi}_c$  is surjective in this case since its target is zero. With surjectivity of  $\bar{\pi}_c$  established in all cases, the equality  $\alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j))$  follows from a diagram chase in (8.14). This concludes the proof of the lemma.  $\square$

The next proposition requires the use of Lemma 8.10 and the results of Section 7.

**Proposition 8.11.** *Fix  $j \in \{0, \dots, p\}$ . If  $c \in (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j)$  is a regular value for the height function  $h$ , then the submodule  $V_j^t|_c \leq H_n(W_t^i|_c)$  is a Lagrangian subspace. Furthermore, for all other  $k = 0, \dots, p$ , we have  $V_k^t|_c \leq V_j^t|_c$ .*

*Proof.* Let  $c \in (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j)$  is a regular value for the height function  $h$ . Proving that  $V_j^t|_c$  is Lagrangian breaks down into two cases depending on the form that the level set  $W_t^i|_c$  takes, according to the two scenarios in Proposition 8.8. The first case is the case where  $W_t^i|_c \cong W|_c, \text{ rel } W'|_c$  and the second is the case when  $W_t^i|_c$  is obtained from  $W|_c$  by a sequence of surgeries in degree  $l$ .

**Case (a):** There is a diffeomorphism  $W_t^i|_c \cong W|_c, \text{ rel } W'|_c$ . Since the diffeomorphism is relative to  $W'$  we obtain a commutative diagram

$$\begin{array}{ccc} H_n(W|_c) & \xleftarrow{\alpha_c} & H_n(W'|_c) \\ & \searrow \varphi \cong & \downarrow \beta_c^t \\ & & H_n(W_t^i|_c), \end{array}$$

where the diagonal map  $\varphi$  is the isomorphism induced by the diffeomorphism

$$W_t^i|_c \cong W|_c, \text{ rel } W'|_c.$$

Since  $\varphi$  is an isomorphism that preserves the intersection form on  $H_n(W_t^i|_c)$ , it follows that

$$\varphi(V_j|_c) \leq H_n(W_t^i|_c)$$

is a Lagrangian subspace. Now,  $\alpha_c$  maps surjectively onto the subspace  $V_j|_c$ . This fact together with commutativity of the above diagram implies that  $\beta_c^t(\alpha_c^{-1}(V_j|_c)) = \varphi(V_j|_c)$ . By Lemma 8.10 we have  $\beta_c^t(\alpha_c^{-1}(V_j|_c)) = V_j^t|_c$ , and thus  $V_j^t|_c$  is a Lagrangian as well. This establishes case (a).

**Case (b):** The manifold  $W_t^i|_c$  is obtained from  $W|_c$  by a sequence of surgeries in degree  $l$ . In this case, Propositions 7.1 or 7.4 (depending on whether  $l < n - 1$  or  $l = n - 1$ ) imply that the subspace  $\beta_c^t(\alpha_c^{-1}(V_j|_c)) \leq H_n(W_t^i|_c)$  is Lagrangian. Again, by Lemma 8.10 we have

$$\beta_c^t(\alpha_c^{-1}(V_j|_c)) = V_j^t|_c,$$

and thus  $V_j^t|_c$  is Lagrangian. This establishes case (b).

Now let  $c \in (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j)$  be a regular value. By combining cases (a) and (b) we have proven that  $V_j^t|_c \leq H_n(W_t^i|_c)$  is a Lagrangian subspace. We still need to show that  $V_k^t|_c \leq V_j^t|_c$  for  $k = 0, \dots, p$ . By definition of  $\mathbf{X}_{\bullet}^{\mathcal{L}, l}$ , we have  $V_k|_c \leq V_j|_c$ . Thus, for all  $t$  we have

$$\beta_c^t(\alpha_c^{-1}(V_k|_c)) \leq \beta_c^t(\alpha_c^{-1}(V_j|_c)).$$

The proof of the statement then follows from Lemma 8.10.  $\square$

It is essential that the pairs  $(W_t^i|_{[a,b]}, W_t^i|_b)$  are  $(n-1)$ -connected for all  $t \in [0, 1]$  and for all pairs of regular values  $a < b \in \bigcup_{j=0}^p (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j)$ . The following proposition is [9, Lemma 3.7].

**Proposition 8.12.** *For any two regular values  $a < b \in \bigcup_{j=0}^p (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j)$  of the height function  $h : W_t^i \rightarrow \mathbb{R}$ , the pair  $(W_t^i|_{[a,b]}, W_t^i|_b)$  is  $(n-1)$ -connected.*

We will need to iterate the construction from (9.5). For what follows consider

$$(a, \varepsilon, (W, \ell_W), V) \in \mathbf{D}_p^{\mathcal{L}, l-1} \quad \text{and} \quad (\Lambda, \delta, e, \ell) \in \mathbf{Y}_0^l(x).$$

For each tuple  $t = (t_0, \dots, t_p) \in [0, 1]^{p+1}$ , we form the element

$$(8.15) \quad \mathcal{K}_{e, \ell}^t(W, \ell_W) = \mathcal{K}_{e_p, \ell_p}^{t_p} \circ \dots \circ \mathcal{K}_{e_0, \ell_0}^{t_0}(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^{\infty-1})$$

by iterating Construction 8.1. Since the embeddings  $e_0, \dots, e_p$  are pairwise disjoint, the above formula does indeed make sense and it also follows that one could permute the order of the above composition and still obtain the same resulting family of manifolds. For  $t = (t_0, \dots, t_p) \in [0, 1]^{p+1}$ , let  $(W_t, \ell_t) = \mathcal{K}_{e, \ell}^t(W, \ell_W)$  and let  $V^t = (V_0^t, \dots, V_p^t)$  be the tuple of subspaces of  $H_{n+1}^{\text{cpt}}(W_t|_{(a_0 - \varepsilon_p, a_p + \varepsilon_p)})$  defined in the same way as (8.6), by iterating Construction 8.1. The corollary below follows by assembling all of the results proven above (Propositions 8.8 - 8.12).

**Corollary 8.13.** *The tuple  $(a, \frac{1}{2}\varepsilon, W_t, \ell_t, V^t)$  has the following properties:*

- For all  $t = (t_0, \dots, t_p) \in [0, 1]^{p+1}$ , the tuple  $(a, \frac{1}{2}\varepsilon, W_t, \ell_t, V_t)$  is an element of  $\mathbf{X}_p^{\mathcal{L}, l-1}$ .
- If  $t_i = 1$ , then for each regular value  $b \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ , the manifold  $W_t|_b$  is  $l$ -connected and the subspace  $V_i^t|_b \leq H^n(W_t)$  is a Lagrangian subspace.
- If  $t = (1, \dots, 1)$  and  $b \in \bigcup_{i=0}^p (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$  is a regular value of the height function, then  $(a, \frac{1}{2}\varepsilon, W_t, \ell_t, V^t)$  is an element of the subspace  $\mathbf{X}_p^{\mathcal{L}, l} \subset \mathbf{X}_p^{\mathcal{L}, l-1}$ .

**8.4. Proof of Theorem 4.11.** We are now in a position to prove the weak homotopy equivalence  $|\mathbf{D}_\bullet^{\mathcal{L}, l-1}| \simeq |\mathbf{D}_\bullet^{\mathcal{L}, l}|$ . By Theorem 8.6 we have weak homotopy equivalences

$$|\mathbf{D}_{\bullet, 0}^{\mathcal{L}, l}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet, \bullet}^{\mathcal{L}, l}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet, -1}^{\mathcal{L}, l}| = |\mathbf{D}_\bullet^{\mathcal{L}, l-1}|,$$

where the first map is induced by inclusion of zero-simplices and the second is induced by the augmentation. Using Construction 8.1 and its iterated version from (9.10), we define a map

$$\mathcal{F}_p : [0, 1]^{p+1} \times \mathbf{D}_{p, 0}^{\mathcal{L}, l} \longrightarrow \mathbf{X}_p^{\mathcal{L}, l-1}, \quad (t, (a, \varepsilon, (W, \ell_W), V), e, \ell) \mapsto (a, \frac{1}{2}\varepsilon, \mathcal{K}_{e, \ell}^t(W, \ell_W), V^t),$$

which has the desired range by Corollary 9.14. Furthermore, Corollary 8.13 implies that  $\mathcal{F}_p$  sends  $(1, \dots, 1) \times \widehat{\mathbf{D}}_{p, 0}^{\mathcal{L}, l}$  to the subspace  $\mathbf{X}_p^{\mathcal{L}, l} \subset \mathbf{X}_p^{\mathcal{L}, l-1}$ . Proposition 8.7 implies that

$$\mathcal{K}_{e_i, \ell_i}^0(W, \ell_W) = (W, \ell_W)$$

for all  $i = 0, \dots, p$ . We obtain the formula

$$d_i \mathcal{F}_p(d^i t, x) = \mathcal{F}_{p-1}(t, d_i x)$$

where  $d^i : [0, 1]^p \rightarrow [0, 1]^{p+1}$  is the embedding that adds a zero in the  $i$ th coordinate and  $d_i$  is the  $i$ th face map associated to the semi-simplicial space  $\mathbf{X}_\bullet^{\mathcal{L}, l-1}$ . We wish to assemble the maps  $\mathcal{F}_p$  to a homotopy  $\mathcal{F} : [0, 1] \times |\widehat{\mathbf{D}}_{p,0}^{\mathcal{L}, l}| \rightarrow |\mathbf{X}_p^{\mathcal{L}, l-1}|$ . By following [9, p. 312 -313] we obtain the proposition below.

**Proposition 8.14.** *The maps  $\mathcal{F}_p$  induce a homotopy  $\mathcal{F} : [0, 1] \times |\mathbf{D}_{\bullet,0}^{\mathcal{L}, l}| \rightarrow |\mathbf{X}_\bullet^{\mathcal{L}, l-1}|$  with the following properties:*

- (i) *The map  $\mathcal{F}(0, \_)$  is equal to  $|\mathbf{D}_{\bullet,0}^{\mathcal{L}, l}| \rightarrow |\mathbf{D}_{\bullet,\bullet}^{\mathcal{L}, l}| \rightarrow |\mathbf{D}_{\bullet,-1}^{\mathcal{L}, l}| \rightarrow |\mathbf{X}_\bullet^{\mathcal{L}, l-1}|$ , which is a weak homotopy equivalence by combination of Theorem 8.6 and Proposition 6.13;*
- (ii) *The map  $\mathcal{F}(1, \_)$  factors through the inclusion  $|\mathbf{X}_\bullet^{\mathcal{L}, l}| \hookrightarrow |\mathbf{X}_\bullet^{\mathcal{L}, l-1}|$ .*

Observe that if  $x = (a, \varepsilon, (W, \ell_W), V) \in \mathbf{D}_p^{\mathcal{L}, l-1}$  is contained in the subspace  $\mathbf{D}_p^{\mathcal{L}, l} \subset \mathbf{D}_p^{\mathcal{L}, l-1}$ , then the *empty surgery data* determined by setting  $\Lambda = \emptyset$  yields a valid element of  $\mathbf{Y}_0(x)$ . Thus, we obtain a semi-simplicial map  $\mathbf{D}_p^{\mathcal{L}, l} \rightarrow \mathbf{D}_{p,0}^{\mathcal{L}, l}$ ,  $x \mapsto (x, \emptyset)$ , and an embedding  $|\mathbf{D}_\bullet^{\mathcal{L}, l}| \hookrightarrow |\mathbf{D}_{\bullet,0}^{\mathcal{L}, l}|$ . Using the homotopy  $\mathcal{F}$  we obtain the diagram

$$\begin{array}{ccc} |\mathbf{D}_\bullet^{\mathcal{L}, l}| & \hookrightarrow & |\mathbf{D}_{\bullet,0}^{\mathcal{L}, l}| \\ \simeq \downarrow & \swarrow \mathcal{F}(1, \_) & \searrow \mathcal{F}(0, \_) \downarrow \simeq \\ |\mathbf{X}_\bullet^{\mathcal{L}, l}| & \hookrightarrow & |\mathbf{X}_\bullet^{\mathcal{L}, l-1}| \end{array}$$

where the upper-triangle is commutative and the lower-triangle is homotopy commutative. It follows from this diagram that the inclusion  $|\mathbf{X}_\bullet^{\mathcal{L}, l}| \hookrightarrow |\mathbf{X}_\bullet^{\mathcal{L}, l-1}|$  is a weak homotopy equivalence. This completes the proof of Theorem 4.11 thus establishing the weak homotopy equivalence  $B\mathbf{Cob}_\theta^{\mathcal{L}, l} \simeq B\mathbf{Cob}_\theta^{\mathcal{L}, l-1}$ .

## 9. SURGERY ON OBJECTS IN THE MIDDLE DIMENSION

In this section we prove Theorem 4.12 which asserts that there are weak homotopy equivalences  $B\mathbf{Cob}_\theta^{\mathcal{L}, n} \xrightarrow{\simeq} B\mathbf{Cob}_\theta^{\mathcal{L}, n-1}$  whenever  $n \geq 4$  and  $n \neq 7$ , the tangential structure  $\theta : B \rightarrow BO(2n+1)$  is weakly once-stable, and the space  $B$  is  $n$ -connected. This section has essentially the same formal outline as Section 8 but the details are different.

**9.1. The surgery move.** We will need to use the surgery move from [9, Section 5.2]. The move is similar to the one that we used in Section 8.1 but has a few key differences (see [9, Page 315] for a discussion of how this surgery move works). Before stating our main proposition, we will need to fix some notation. Let  $K^n \subset \mathbb{R}^{n+1} \times D^{n+1}$  be the same submanifold from (8.1). The surgery move

will require us to add one more coordinate to the background space. Let  $K_0^n \subset \mathbb{R} \times \mathbb{R}^{n+1} \times D^{n+1}$  be the submanifold given by

$$(9.1) \quad K_0^n := \{0\} \times K^n \subset \mathbb{R} \times \mathbb{R}^{n+1} \times D^{n+1}.$$

We let  $h : K_0^n \rightarrow \mathbb{R}$  be the function given by projecting onto the second coordinate in the background space  $\mathbb{R} \times \mathbb{R}^{n+1} \times D^{n+1}$  (in other words  $h$  is defined by projecting onto the first coordinate of  $\mathbb{R}^{n+1} \times D^{n+1}$ ). Let  $\ell$  be a  $\theta$ -structure on  $K_0^n$ . We define

$$(9.2) \quad (\mathcal{P}_t^n, \ell_t) \in \Psi_\theta(\mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times \mathbb{R}^{n+1}), \quad t \in [0, 1],$$

to be the one-parameter family of manifolds constructed in [9, Section 5.2], with initial value given by

$$(\mathcal{P}_0^n, \ell_0) = (K_0^n|_{(-6, -2)}, \ell|_{(-6, -2)}).$$

By their construction, the correspondence  $\ell \mapsto (\mathcal{P}_t^n, \ell_t)$  defines a continuous map

$$\text{Bun}(TK^n, \theta^* \gamma^{2n+1}) \rightarrow \text{Maps}([0, 1], \Psi_\theta(\mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times \mathbb{R}^{n+1})).$$

**Remark 9.1.** We emphasize that this one parameter family is different from the one used in Section 8.1. In particular it is essential to the construction that the ambient space have an extra factor of  $\mathbb{R}$  in its first coordinate. We refer the reader to [9, Pages 314-316] for a discussion of its basic properties.

The proposition below sums up the main properties of the one-parameter family (8.2) and is essentially a restatement of [9, Proposition 5.2]. This will be used extensively in Section 9.3.

**Proposition 9.2.** *Let  $\ell$  be a  $\theta$ -structure on  $K_0^n$ . Then the one parameter family from (9.2)*

$$(\mathcal{P}_t^n, \ell_t) \in \Psi_\theta(\mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times \mathbb{R}^{n+1}), \quad t \in [0, 1],$$

*satisfies the following conditions:*

(i)  $(\mathcal{P}_0^n, \ell_0) = (K_0^n|_{(-6, -2)}, \ell|_{(-6, -2)}).$

(ii) *For all  $t$ ,  $(\mathcal{P}_t^n, \ell_t)$  agrees with  $(K_0^n|_{(-6, -2)}, \ell|_{(-6, -2)})$  outside of the region*

$$(-2, 2) \times (-5, -2) \times B_2^n(0) \times D^{n+1}.$$

*We will denote  $\mathcal{P}^\partial := K_0^n|_{(-6, -2)} \setminus [(-2, 2) \times (-5, -2) \times B_2^n(0) \times D^{n+1}]$ .*

(iii) *For all  $t$  and each pair of regular values  $-6 < a < b < -2$  of the height function  $h : \mathcal{P}_t^n \rightarrow \mathbb{R}$ , the pair  $(\mathcal{P}_t^n|_{[a, b]}, \mathcal{P}_t^n|_b)$  is  $(n-1)$ -connected.*

(iv) *Let  $a$  be a regular value for the height function  $h : \mathcal{P}_t^n \rightarrow (-6, -2)$ :*

(a) *If  $a$  is outside of  $(-4, -3)$  then there is a diffeomorphism*

$$\mathcal{P}_t^n|_a \cong \mathcal{P}_0^n|_a \text{ rel } \mathcal{P}^\partial|_a.$$

(b) *If  $a$  is inside of  $(-4, -3)$  then either one of the following two scenarios apply:*

- there is a diffeomorphism  $\mathcal{P}_t^n|_a \cong \mathcal{P}_0^n|_a \text{ rel } \mathcal{P}^\partial|_a$ , or
- $\mathcal{P}_t^n|_a$  is obtained from  $\mathcal{P}_0^n|_a$  by surgery along the standard embedding.

$$D^n \times S^n \hookrightarrow \{0\} \times \{a\} \times \mathbb{R}^n \times S^n = \mathcal{P}_0^n|_a,$$

(v) The critical values of  $h : \mathcal{P}_1^n \longrightarrow (-6, -2)$  are  $-4$  and  $-3$ . For  $a \in (-4, -3)$ , the manifold  $\mathcal{P}_1^n|_a$  is obtained from  $\mathcal{P}_0^n|_a$  by surgery along the standard embedding

$$D^n \times S^n \hookrightarrow \{0\} \times \{a\} \times \mathbb{R}^n \times S^n = \mathcal{P}_0^n|_a.$$

**9.2. A semi-simplicial resolution.** Below we construct an augmented bi-semi-simplicial space similar to the one from Section 8.2. First we must choose once and for all, smoothly in the data  $(a_i, \varepsilon_i, a_p, \varepsilon_p)$ , increasing diffeomorphisms

$$(9.3) \quad \psi = \psi(a_i, \varepsilon_i, a_p, \varepsilon_p) : (-6, -2) \xrightarrow{\cong} (a_i - \varepsilon_i, a_p + \varepsilon_p),$$

sending  $[-4, -3]$  linearly onto  $[a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i]$ . Recall from (9.1) the manifold

$$K_0^n \subset \mathbb{R} \times \mathbb{R}^{n+1} \times D^{n+1}.$$

**Definition 9.3.** Fix once and for all an infinite set  $\Omega$ . Let  $x = (a, \varepsilon, (W, \ell_W), V) \in \mathbf{D}_p^{\mathcal{L}, n-1}$ . With  $a = (a_0, \dots, a_p)$ , for each  $i$  we write  $M_i$  for the manifold  $W|_{a_i}$ . Define  $\mathbf{Y}_0^n(x)$  to consist of tuples  $(\Lambda, \delta, e, \ell)$ , where:

- $\Lambda \subset \Omega$  is a finite subset;
- $\delta : \Lambda \longrightarrow [p]$  is a function;
- $e : \Lambda \times \mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times D^{n+1} \longrightarrow \mathbb{R} \times (-1, 1)^{\infty-1}$  is an embedding;
- $\ell$  is a  $\theta$ -structure on  $\Lambda \times K$ .

For  $i \in [p]$ , we write  $\Lambda_i = \delta^{-1}(i)$  and

$$e_i : \Lambda_i \times \mathbb{R} \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^n \times D^{n+1} \longrightarrow \mathbb{R} \times (-1, 1)^{\infty-1}$$

for the embedding obtained by restricting  $e$  and reparametrising using (9.3). We let  $\ell_i$  denote the restriction of  $\ell$  to  $\Lambda_i \times K|_{(-6, -2)}$ . This data is required to satisfy the following conditions:

(i)  $e^{-1}(W) = \Lambda \times \{0\} \times (-6, -2) \times \mathbb{R}^n \times \partial D^{n+1}$ . We let

$$\partial e : \Lambda \times \{0\} \times (-6, -2) \times \mathbb{R}^n \times \partial D^{n+1} \longrightarrow W$$

denote the embedding restricted to the boundary.

(ii) Let  $i \in [p]$ . For  $t \in \bigcup_{k=i}^p (a_k - \varepsilon_k, a_k + \varepsilon_k)$ , we require

$$(x_1 \circ e_i)^{-1}(t) = \Lambda_i \times \mathbb{R} \times \{t\} \times \mathbb{R}^n \times D^{n+1}.$$

In other words,  $e_i$  is height-preserving over the intervals  $(a_k - \varepsilon_k, a_k + \varepsilon_k)$  for  $k = i, \dots, p$ .

(iii) The restriction of  $\ell$  to  $\Lambda \times K_0^n|_{(-6,-2)}$  agrees with the composition

$$\ell_W \circ D(\partial e) : T(\Lambda \times K_0^n|_{(-6,-2)}) \longrightarrow \theta^* \gamma.$$

(iv) Let  $W' \subset W|_{(a_0-\varepsilon_0, a_p+\varepsilon_p)}$  denote the complement  $W|_{(a_0-\varepsilon_0, a_p+\varepsilon_p)} \setminus \text{Im}(e)$  and let

$$\alpha : H_{n+1}^{\text{cpt}}(W') \longrightarrow H_{n+1}^{\text{cpt}}(W|_{(a_0-\varepsilon_0, a_p+\varepsilon_p)})$$

denote the map induced by inclusion. We require that the subspace

$$\sum_{i=0}^p V_i \leq H_{n+1}^{\text{cpt}}(W|_{(a_0-\varepsilon_0, a_p+\varepsilon_p)})$$

be contained in the image of the map  $\alpha$ .

(See the Lemma 9.4 below for an explanation of condition iv.) For each pair  $i \in [p]$ , the restriction of  $e$  to  $\Lambda_i \times \{0\} \times \{a_i\} \times \mathbb{R}^n \times \partial D^{n+1}$  yields the embedding

$$(9.4) \quad \partial e_i : \Lambda_i \times \{0\} \times \{a_i\} \times \mathbb{R}^n \times \partial D^{n+1} \longrightarrow M_i.$$

Let  $\widetilde{M}_i$  denote the manifold obtained by performing surgery on  $M_i$  along the embedding  $(\partial e_i, \ell_i)$ . The final condition is:

(v) For each  $i \in [p]$  the resulting manifold  $\widetilde{M}_i$  is  $n$ -connected, or in other words is a homotopy sphere.

**Lemma 9.4.** *For each  $\lambda \in \Lambda_i$ , the homology class represented by the embedding*

$$\partial e_i^\lambda : \{\lambda\} \times \{0\} \times \{a_i\} \times \{0\} \times \partial D^{n+1} \longrightarrow M_i$$

*(obtained by restricting  $\partial e_i$ ) is contained in the Lagrangian subspace  $V_i|_{M_i} \leq H_n(M_i)$ .*

*Proof.* By 7.2 we have an exact sequence

$$H_n(M'_i) \longrightarrow H_n(M_i) \xrightarrow{\lambda(\cdot, [\partial e_i^\lambda])} \mathbb{Z}.$$

Therefore  $[\partial e_i^\lambda]$  pairs trivially with the entire image of  $H_n(M'_i)$ , in particular with  $V_i|_{M_i}$  by condition iv). Since  $V_i|_{M_i}$  is a lagrangian we find  $[\partial e_i^\lambda] \in V_i|_{M_i}$ .  $\square$

As before we use the above construction to define a bi-semi-simplicial space.

**Definition 9.5.** For  $q \in \mathbb{Z}_{\geq 0}$  we define  $\mathbf{Y}_q^n(x) \subset \mathbf{Y}_0^n(x)^{q+1}$  to be the subset consisting of those  $(q+1)$ -tuples

$$((\Lambda^0, \delta^0, e^0, \ell^0), \dots, (\Lambda^q, \delta^q, e^q, \ell^q)) \in (\mathbf{Y}_0^n(x))^{\times(q+1)}$$

that such that  $\text{Im}(e^i) \cap \text{Im}(e^j) = \emptyset$  whenever  $i \neq j$ , where  $\text{Im}(e_i)$  denotes the image of  $e_i$ . We then define

$$\mathbf{D}_{p,q}^{\mathcal{L},n} = \{(x, y) \mid x \in \mathbf{D}_p^{\mathcal{L},n-1}, y \in \mathbf{Y}_q(x)\},$$

and the correspondence  $[p, q] \mapsto \mathbf{D}_{p,q}^{\mathcal{L},n}$  defines a bi-semi-simplicial space  $\mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n}$ . We topologize each  $\mathbf{D}_{p,q}^{\mathcal{L},n}$  in the same way as in Definition 8.5.

For each  $q$  we have a projection  $\mathbf{D}_{p,q}^{\mathcal{L},n} \longrightarrow \mathbf{D}_p^{\mathcal{L},n-1}$  which yields an augmented bi-semi-simplicial space  $\mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n} \longrightarrow \mathbf{D}_{\bullet,-1}^{\mathcal{L},n}$ , with  $\mathbf{D}_{\bullet,-1}^{\mathcal{L},n} = \mathbf{D}_{\bullet}^{\mathcal{L},n-1}$ . The theorem stated below will be proven in Section 10.

**Theorem 9.6.** *Let  $n \geq 4$ . Then there are weak homotopy equivalences*

$$|\mathbf{D}_{\bullet,0}^{\mathcal{L},n}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet,-1}^{\mathcal{L},n}| = |\mathbf{D}_{\bullet}^{\mathcal{L},n-1}|,$$

where the first map is induced by inclusion of zero-simplices and the second is induced by the augmentation.

**Remark 9.7.** The proof of the above theorem is formally similar to the proof of [9, Theorem 5.14] but the geometric details are different. In particular, in the proof of the above theorem the Lagrangian subspaces associated to the elements of  $\mathbf{D}_{\bullet}^{\mathcal{L},n-1}$  will play a crucial role in the proof.

**9.3. Implementation of the surgery move.** We now show how to implement the surgery move that was constructed in Section 9.1. Again recall how to use this surgery move to define a semi-simplicial map  $\mathbf{D}_{\bullet,0}^{\mathcal{L},n} \longrightarrow \mathbf{X}_{\bullet}^{\mathcal{L},n-1}$ . The construction is similar to what was done in Construction 8.1. For what follows let

$$x = (a, \varepsilon, (W, \ell_W), V) \in \mathbf{D}_p^{\mathcal{L},n-1} \quad \text{and} \quad y = (\Lambda, \delta, e, \ell) \in \mathbf{Y}_0^n(x).$$

In this way the pair  $(x, y)$  is an element of the space  $\mathbf{D}_{p,0}^{\mathcal{L},n}$ . For each  $i = 0, \dots, p$ , we have an embedding  $e_i$  and a  $\theta$ -structure  $\ell_i$  on  $\Lambda_i \times K_0^n$ , where  $\Lambda_i = \delta^{-1}(i)$ . From this data we may construct a one-parameter family of  $\theta$ -manifolds

$$(9.5) \quad \mathcal{K}_{e_i, \ell_i}^t(W, \ell_W) \in \Psi_{\theta}((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^{\infty-1}).$$

We define  $\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W)$  to be equal to  $W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}$  outside the image of  $e_i$ , and on the image of  $e_i$  it is given by  $e_i(\Lambda_i \times \bar{\mathcal{P}}_t^n)$ .

We transport the subspaces  $V_0, \dots, V_p \leq H_{n+1}^{\text{cpt}}(W)$  over to  $H_{n+1}^{\text{cpt}}(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W))$  in the same way as in Construction 8.1 as well: Let  $W'$  denote  $W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p) \setminus \text{Im}(e_i)}$ . For each  $t \in [0, 1]$ ,

$$H_{n+1}^{\text{cpt}}(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}) \xleftarrow{\alpha} H_{n+1}^{\text{cpt}}(W') \xrightarrow{\beta^t} H_{n+1}^{\text{cpt}}(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W))$$

For each  $j = 0, \dots, p$  we let

$$(9.6) \quad V_j^t \leq H_{n+1}^{\text{cpt}}(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W))$$

be the subspace given by

$$\beta_t(\alpha^{-1}(V_j)) \leq H_{n+1}^{\text{cpt}}(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W)).$$

The following proposition is analogous to Proposition 8.7.



**Proposition 9.8.** *The above construction defines a one-parameter family*

$$(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W), V_j^t) \in \Psi_\theta^\Delta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty), \quad t \in [0, 1],$$

with initial value given by

$$(\mathcal{K}_{e_i, \ell_i}^0(W, \ell_W), V_j^0) = (W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, \ell|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}).$$

*Proof.* The condition

$$\mathcal{K}_{e_i, \ell_i}^0(W, \ell_W) = (W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, \ell|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})$$

follows immediately from the construction. As with the proof of Proposition 8.7, to prove that  $V_j^0 = V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}$  it will suffice to show that  $\alpha$  maps surjectively onto the subspace  $V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}$ . This follows from Definition 9.3 (condition (iv)). This establishes the statement about the initial value of  $(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W), V_j^t)$ . The continuity of  $(\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W), V_j^t)$  follow directly from Proposition 2.12.  $\square$

Again, we make some basic observations about the above construction, being brief when lagrangians are not involved. Fix  $i \in \{0, \dots, p\}$  and let  $(W_t^i, \ell_t^i)$  denote the family of  $\theta$ -manifolds  $\mathcal{K}_{e_i, \ell_i}^t(W, \ell_W)$  defined in (9.5). For each  $j \in \{0, \dots, p\}$ ,  $V_j^t$  is the subspace of  $H_{n+1}^{\text{cpt}}(W_t)$  defined in (9.6). We let  $W'$  denote the complement  $W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)} \setminus \text{Im}(e_i)$ . As before, all constructions take place inside  $(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty$  and so we will abuse notation and write

$$(9.7) \quad W := W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}.$$

Propositions 9.9 - 9.13 all use the same one-parameter family  $W_t^i$  defined above with the same choice of  $i$  fixed throughout. Both of the following propositions are immediate consequences of Proposition 9.2, part (iv).

**Proposition 9.9.** *Let  $j \neq i$ , and let  $c \in (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j)$  be a regular value for the height function  $h : W_t^i \rightarrow \mathbb{R}$ . Then  $c$  is also a regular value for  $h : W \rightarrow \mathbb{R}$  and there is a diffeomorphism  $W_t^i|_c \cong W|_c$ , rel  $W'|_c$ .*

The next proposition records the behavior of the level sets  $W_t^i|_c$  in the case that the regular value  $c$  lies in the interval  $(a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ .

**Proposition 9.10.** *Let  $c \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$  be a regular value for  $h : W_t^i \rightarrow \mathbb{R}$ . Then  $c$  is also a regular value for  $h : W \rightarrow \mathbb{R}$  and one of the following two scenarios holds:*

- (a) *there is a diffeomorphism  $W_t^i|_c \cong W|_c$ , rel  $W'|_c$ , or*
- (b)  *$W_t^i|_c$  is  $n$ -connected and is obtained from  $W|_c$  by a collection of  $n$ -surgeries.*

*Furthermore if  $t = 1$  then scenario (b) always holds for the level set  $W_1^i|_c$ .*

The following lemma is similar to Lemma 8.10 and in fact, it is formally the exact same statement.

**Lemma 9.11.** *Let  $c \in \cup_{k=0}^p (a_k - \varepsilon_k, a_k + \varepsilon_k)$  be a regular value for the height function  $h : W_t^i \rightarrow \mathbb{R}$ . Let*

$$H_n(W|_c) \xleftarrow{\alpha_c} H_n(W'|_c) \xrightarrow{\beta_c^t} H_n(W_t^i|_c)$$

*denote the maps induced by inclusion. Then for any  $j = 0, \dots, p$ , the two subspaces*

$$V_j^t|_c \leq H_n(W_t^i|_c) \quad \text{and} \quad \beta_c^t(\alpha_c^{-1}(V_j|_c)) \leq H_n(W_t^i|_c)$$

*are equal.*

*Proof.* Consider the commutative diagram

$$(9.8) \quad \begin{array}{ccc} H_{n+1}^{\text{cpt}}(W) & \xleftarrow{\alpha} & H_{n+1}^{\text{cpt}}(W') \\ \downarrow \pi_c & & \downarrow \pi'_c \\ H_n(W|_c) & \xleftarrow{\alpha_c} & H_n(W'|_c). \end{array}$$

As in the proof of Lemma 8.10, it will suffice to prove the equality  $\alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j))$ . As in the proof of Lemma 8.10, let  $P_i$  and  $P_i^\partial$  be the manifolds

$$(9.9) \quad \begin{aligned} P_i &= \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times S^n \times \mathbb{R}^n, \\ P_i^\partial &= \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times S^n \times (\mathbb{R}^n \setminus D^n). \end{aligned}$$

As before, we have excision isomorphisms

$$\begin{aligned} H_k^{\text{cpt}}(W, W') &\cong H_k^{\text{cpt}}(P_i, P_i^\partial) \\ H_k(W|_c, W'|_c) &\cong \begin{cases} H_k^{\text{cpt}}(P_i|_c, P_i^\partial|_c) & \text{if } c \in (a_i - \varepsilon_i, a_p + \varepsilon_p), \\ 0 & \text{if } c \notin (a_i - \varepsilon_i, a_p + \varepsilon_p). \end{cases} \end{aligned}$$

It follows that  $H_*^{\text{cpt}}(W, W')$  is trivial in all degrees other than  $(n+1)$  and that  $H_k(W|_c, W'|_c)$  is trivial in all degrees other than  $n$  (if  $c \notin (a_i - \varepsilon_i, a_p + \varepsilon_p)$  then  $H_k(W|_c, W'|_c)$  is trivial in all degrees). Using this together with the long exact sequences on  $H_*^{\text{cpt}}$  and  $H_*$  associated to the pairs  $(W, W')$  and  $(W|_c, W'|_c)$ , it follows that both maps  $\alpha$  and  $\alpha_c$  are injective. By part (iv) of Definition 9.3, every element of  $V_j$  lies in the image of  $\alpha$ . Using these facts, a simple diagram chase in (9.8) proves that  $\alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j))$ . This concludes the proof of the lemma.  $\square$

The next proposition is similar to Proposition 8.11 using Lemma 9.11.

**Proposition 9.12.** *Let  $j = 0, \dots, p$ . If  $c \in (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j)$  is a regular value for the height function  $h$ , then the submodule  $V_j^t|_c \leq H_n(W_t^i|_c)$  is a Lagrangian subspace. Furthermore,  $V_k^t|_c \leq V_j^t|_c$  for  $k = 0, \dots, p$ .*

*Proof.* By Proposition 9.9 there are two cases depending on the form that the level set  $W_t^i|_c$  takes.

**Case (a):** There is a diffeomorphism  $W_t^i|_c \cong W|_c, \text{ rel } W'|_c$ . Since the diffeomorphism is relative to  $W'$ , as in the proof of Proposition 8.11 we obtain a commutative diagram

$$\begin{array}{ccc} H_n(W|_c) & \xleftarrow{\alpha_c} & H_n(W'|_c) \\ & \searrow \varphi & \downarrow \beta_c^t \\ & & H_n(W_t^i|_c), \end{array}$$

where the diagonal map  $\varphi$  is the isomorphism induced by the  $\theta$ -diffeomorphism

$$W_t^i|_c \cong W|_c, \text{ rel } W'|_c.$$

Now using the fact that  $\varphi$  preserves both intersection and selfintersection pairing and  $V_j|_c$  lies in the image of  $\alpha_c$  (condition (iv) of 9.3 again), the result follows from another diagram chase.

**Case (b):** The manifold  $W_t^i|_c$  is obtained from  $W|_c$  by a collection of surgeries in degree  $n$  and  $W_t^i|_c$  is  $n$ -connected. It follows that  $H_n(W_t^i|_c) = 0$  and thus  $V_j^t|_c$  is automatically Lagrangian.

The property that  $V_k^t|_c \leq V_j^t|_c$  for  $k = 0, \dots, p$  follows in the same way as in the proof of Proposition 8.11. This concludes the proof of the proposition.  $\square$

The next check to be done is that our construction preserves the connectivity of the cobordisms relative to their outgoing boundaries. This is proven in the same way as [9, Lemma 3.6]. We omit the proof.

**Proposition 9.13.** *For any two regular values  $a < b \in \bigcup_{j=0}^p (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j)$  of the height function  $h : W_t^i \rightarrow \mathbb{R}$ , the pair  $(W_t^i|_{[a,b]}, W_t^i|_b)$  is  $(n-1)$ -connected.*

We will need to iterate the construction from (9.5). Consider elements

$$(a, \varepsilon, (W, \ell_W), V) \in \mathbf{D}_p^{\mathcal{L}, n-1} \quad \text{and} \quad (\Lambda, \delta, e, \ell) \in \mathbf{Y}_0(x).$$

For each tuple  $t = (t_0, \dots, t_p) \in [0, 1]^{p+1}$ , we form the element

$$(9.10) \quad \mathcal{K}_{e, \ell}^t(W, \ell_W) = \mathcal{K}_{e_p, \ell_p}^{t_p} \circ \dots \circ \mathcal{K}_{e_0, \ell_0}^{t_0}(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^{\infty-1})$$

by iterating the above construction. For what follows, for  $t = (t_0, \dots, t_p) \in [0, 1]^{p+1}$  let  $W_t = \mathcal{K}_{e, \ell}^t(W, \ell_W)$  be defined as in (9.10) and let  $V^t = (V_0^t, \dots, V_p^t)$  be defined by iterating (9.6). The corollary stated below follows by assembling together all of the results proven above (Propositions 9.9 - 9.13).

**Corollary 9.14.** *The tuple  $(a, \frac{1}{2}\varepsilon, W_t, V^t)$  has the following properties:*

- For all  $t = (t_0, \dots, t_p) \in [0, 1]^{p+1}$ , the tuple  $(a, \frac{1}{2}\varepsilon, W_t, V^t)$  is an element in  $\mathbf{X}_p^{\mathcal{L}, n-1}$ .
- If  $t_i = 1$ , then for each regular value  $b \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ , the manifold  $W_t|_b$  is  $n$ -connected.
- If  $t = (1, \dots, 1)$  and  $b \in \bigcup_{i=0}^p (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$  is a regular value of the height function  $h : W_t \rightarrow \mathbb{R}$ , then  $(a, \frac{1}{2}\varepsilon, W_t, V^t)$  is an element of the subspace  $\mathbf{X}_p^{\mathcal{L}, n} \subset \mathbf{X}_p^{\mathcal{L}, n-1}$ .

With the above corollary established we are now in a position to prove the weak homotopy equivalence  $|\mathbf{D}_{\bullet}^{\mathcal{L},n-1}| \simeq |\mathbf{D}_{\bullet}^{\mathcal{L},n}|$ , which implies Theorem 4.12. By Theorem 9.6 we have weak homotopy equivalences  $|\mathbf{D}_{\bullet,0}^{\mathcal{L},n}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet,-1}^{\mathcal{L},n}| = |\mathbf{D}_{\bullet}^{\mathcal{L},n-1}|$ , where the first map is induced by inclusion of zero-simplices and the second is induced by the augmentation. Using (9.10) we define a map

$$\mathcal{F}_p : [0, 1]^{p+1} \times \mathbf{D}_{p,0}^{\mathcal{L},n} \longrightarrow \mathbf{X}_p^{\mathcal{L},n-1}, \quad (t, (a, \varepsilon, (W, \ell_W), V, e, \ell)) \mapsto (a, \tfrac{1}{2}\varepsilon, \mathcal{K}_{e,\ell}^t(W, \ell_W), V^t),$$

which has the desired range by Corollary 9.14. Furthermore, Corollary 9.14 implies that  $\mathcal{F}_p$  sends  $(1, \dots, 1) \times \mathbf{D}_{p,0}^{\mathcal{L},n}$  to the subspace  $\mathbf{X}_p^{\mathcal{L},n} \subset \mathbf{X}_p^{\mathcal{L},n-1}$ . By Proposition 9.8 implies that

$$\mathcal{K}_{e_i, \ell_i}^0(W, \ell_W) = (W, \ell_W)$$

for all  $i = 0, \dots, p$ . We thus obtain the formula

$$d_i \mathcal{F}_p(d^i t, x) = \mathcal{F}_{p-1}(t, d_i x)$$

where  $d^i : [0, 1]^p \longrightarrow [0, 1]^{p+1}$  is the embedding that adds a zero in the  $i$ th coordinate and  $d_i$  is the  $i$ th face map associated to the semi-simplicial space  $\mathbf{X}_{\bullet}^{\mathcal{L},n-1}$ . As in Proposition 8.14, the maps  $\mathcal{F}_p$  to a homotopy  $\mathcal{F} : [0, 1] \times |\mathbf{D}_{p,0}^{\mathcal{L},n}| \longrightarrow |\mathbf{X}_p^{\mathcal{L},n-1}|$  with the properties:

- (i) The map  $\mathcal{F}(0, \_)$  is equal to the composite  $|\mathbf{D}_{\bullet,0}^{\mathcal{L},n}| \longrightarrow |\mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n}| \longrightarrow |\mathbf{D}_{\bullet,-1}^{\mathcal{L},n}| \longrightarrow |\mathbf{X}_{\bullet}^{\mathcal{L},n-1}|$ , which is a weak homotopy equivalence by Theorem 9.6 and Proposition 6.13;
- (ii) The map  $\mathcal{F}(1, \_)$  factors through the inclusion  $|\mathbf{X}_{\bullet}^{\mathcal{L},n}| \hookrightarrow |\mathbf{X}_{\bullet}^{\mathcal{L},n-1}|$ .

With this homotopy established, the proof of the weak homotopy equivalence  $|\mathbf{X}_{\bullet}^{\mathcal{L},n}| \simeq |\mathbf{X}_{\bullet}^{\mathcal{L},n-1}|$  follows in exactly same way as in Section 8.4. This completes the proof of Theorem 9.6.

## 10. CONTRACTIBILITY OF THE SPACE OF SURGERY DATA

We proceed with the proof of Theorem 9.6 which asserts that there are weak homotopy equivalences

$$|\mathbf{D}_{\bullet,0}^{\mathcal{L},n}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet,-1}^{\mathcal{L},n}| = |\mathbf{D}_{\bullet}^{\mathcal{L},n-1}|,$$

where the first map is induced by the inclusion of zero-simplices and the second is induced by the augmentation. The first homotopy equivalence  $|\mathbf{D}_{\bullet,0}^{\mathcal{L},n}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n}|$  is proven in exactly the same way as Lemma 6.10 (see also [9, Page 327]) and so we omit the proof of this and focus on establishing the second weak homotopy equivalence,  $|\mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n}| \xrightarrow{\simeq} |\mathbf{D}_{\bullet}^{\mathcal{L},n-1}|$ . We want to apply Theorem 6.6 to each augmented semi-simplicial space  $\mathbf{D}_{p,\bullet}^{\mathcal{L},n} \xrightarrow{\simeq} \mathbf{D}_{p,-1}^{\mathcal{L},n}$ . It turns out that a slightly stronger version of Theorem 6.6 will be needed.

**10.1. A strengthening of Theorem 6.6.** Let  $X_\bullet \rightarrow X_{-1}$  be an augmented topological flag complex. To prove that  $|X_\bullet| \rightarrow X_{-1}$  is a weak homotopy equivalence, it would suffice to verify conditions (i), (ii), and (iii) from Theorem 6.6. In our case, however, condition (iii) for an arbitrary set of vertices  $\{v_1, \dots, v_m\} \subset \varepsilon^{-1}(x)$  does not seem to hold. We will need to use the following version of Theorem 6.6 in which condition (iii) is slightly weakened.

We will need to consider a symmetric relation  $\mathcal{R}$  that is open and dense as a subset of the fibred product  $X_0 \times_{X_{-1}} X_0$ . The proof of the following result is given in [1, Theorem 6.4].

**Theorem 10.1.** *Let  $X_\bullet \rightarrow X_{-1}$  be an augmented topological flag complex that satisfies conditions (i) and (ii) of Theorem 6.6. Let  $\mathcal{R} \subset X_0 \times_{X_{-1}} X_0$  be an open and dense symmetric relation with the property that  $X_1 \subset \mathcal{R}$ . Suppose that  $X_\bullet \rightarrow X_{-1}$  satisfies the following further condition:*

(iii)\* *Let  $x \in X_{-1}$ . Given:*

- *a non-empty subset,  $\{v_1, \dots, v_m\} \subset \varepsilon^{-1}(x)$ , whose elements are pairwise related by  $\mathcal{R}$ , and*
- *an arbitrary subset,  $\{w_1, \dots, w_k\} \subset \varepsilon^{-1}(x)$ , such that  $(v_i, w_j) \in X_1$  for all  $i, j$ ,*

*there exists  $v \in \varepsilon^{-1}(x)$  such that  $(v, v_i) \in X_1$  and  $(v, w_j) \in X_1$  for all  $i, j$ .*

*If condition (iii)\* is satisfied for all  $x \in X_{-1}$  then the induced map  $|X_\bullet| \rightarrow X_{-1}$  is a weak homotopy equivalence.*

**Remark 10.2.** The above theorem is based on [10], *Erratum to: stable moduli spaces of high-dimensional manifolds*; its proof is essentially an abstraction of the techniques from that erratum, see [1, Theorem 6.4].

**10.2. Proof of Theorem 9.6.** We will need to work with a more flexible model for the bi-semi-simplicial space  $\mathbf{D}_{\bullet, \bullet}^{\mathcal{L}, n}$ . Let us define the *core*

$$(10.1) \quad C = \{0\} \times (-6, -2) \times \{0\} \times D^{n+1} \subset \mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times D^{n+1}.$$

**Definition 10.3.** For  $x = (a, \varepsilon, (W, \ell_W), V) \in \mathbf{D}_{\bullet, \bullet}^{\mathcal{L}, n-1}$ , let  $\tilde{\mathbf{Y}}_\bullet(x)$  be the semi-simplicial space defined as in Definition 9.3 except now we only ask the map  $e$  to be a smooth embedding on a neighborhood of the subset,  $\Lambda \times C \subset \Lambda \times (\mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times D^{n+1})$ . Let  $\tilde{\mathbf{D}}_{\bullet, \bullet}^{\mathcal{L}, n}$  be the bi-semi-simplicial space defined by setting  $\tilde{\mathbf{D}}_{p, q}^{\mathcal{L}, n} = \{(x, y) \mid x \in \mathbf{D}_p^{\mathcal{L}, n-1}, y \in \tilde{\mathbf{Y}}_q(x)\}$ . Using the projection we obtain an augmented bi-semi-simplicial space  $\tilde{\mathbf{D}}_{\bullet, \bullet}^{\mathcal{L}, n} \rightarrow \tilde{\mathbf{D}}_{\bullet, -1}^{\mathcal{L}, n}$  with  $\tilde{\mathbf{D}}_{\bullet, -1}^{\mathcal{L}, n} = \tilde{\mathbf{D}}_{\bullet, \bullet}^{\mathcal{L}, n-1}$ .

In view of Theorem 10.1, we define a symmetric relation on the 0-simplices of  $\tilde{\mathbf{D}}_{p, \bullet}^{\mathcal{L}, n}$ , for each  $p \in \mathbb{Z}_{\geq 0}$ .

**Definition 10.4.** Let  $\mathcal{T} \subset \tilde{\mathbf{D}}_{p, 0}^{\mathcal{L}, n} \times_{\tilde{\mathbf{D}}_{p, -1}^{\mathcal{L}, n}} \tilde{\mathbf{D}}_{p, 0}^{\mathcal{L}, n}$  be the subset consisting of those

$$((a, \varepsilon, (W, \ell_W), V), (\Lambda_1, \delta_1, e_1, \ell_1), (\Lambda_2, \delta_2, e_2, \ell_2))$$

such that the embeddings  $e_1|_{\Lambda_1 \times C}$  and  $e_2|_{\Lambda_2 \times C}$  are transverse. This subset  $\mathcal{T}$  is clearly a symmetric and open relation. By the Thom transversality theorem applied to each of the fibres over  $\tilde{\mathbf{D}}_{p,-1}^{\mathcal{L},n}$  we see that it is a dense subset of the fibred product  $\mathcal{T} \subset \tilde{\mathbf{D}}_{p,0}^{\mathcal{L},n} \times_{\tilde{\mathbf{D}}_{p,-1}^{\mathcal{L},n}} \tilde{\mathbf{D}}_{p,0}^{\mathcal{L},n}$ .

Just as in [9, Proposition 6.15] it follows that the inclusion  $\tilde{\mathbf{D}}_{\bullet,\bullet}^{\mathcal{L},n} \hookrightarrow \mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n}$  is a level-wise weak homotopy equivalence and thus it induces a weak homotopy equivalence  $|\tilde{\mathbf{D}}_{\bullet,\bullet}^{\mathcal{L},n}| \simeq |\mathbf{D}_{\bullet,\bullet}^{\mathcal{L},n}|$ . To prove Theorem 9.6 it is enough to prove the weak homotopy equivalence  $|\tilde{\mathbf{D}}_{\bullet,\bullet}^{\mathcal{L},n}| \simeq |\tilde{\mathbf{D}}_{\bullet,-1}^{\mathcal{L},n}|$ . To do this we will show that for each  $p \in \mathbb{Z}_{\geq 0}$ , the augmented topological flag complex

$$(10.2) \quad \tilde{\mathbf{D}}_{p,\bullet}^{\mathcal{L},n} \longrightarrow \tilde{\mathbf{D}}_{p,-1}^{\mathcal{L},n}$$

induces a weak homotopy equivalence  $|\tilde{\mathbf{D}}_{p,\bullet}^{\mathcal{L},n}| \simeq |\tilde{\mathbf{D}}_{p,-1}^{\mathcal{L},n}|$ . With this established, geometrically realizing the first coordinate will yield the theorem. In order to prove that (10.2) induces a weak homotopy equivalence on geometric realization, we will show that it satisfies conditions (i), (ii), and (iii)\* (with respect to the relation  $\mathcal{T}$ , see Definition 10.4) from the statement of Theorem 10.1. Condition (i) follows directly from [9, Proposition 6.10]. In the sections below we prove conditions (ii) and (iii)\*. We begin with condition (iii)\*.

**10.3. Condition (iii)\*.** We will need to develop some preliminary results first. The key technique that will be used is the *higher dimensional half-Whitney trick* from [24, Theorem A.14] and [1]. We will need to use the following lemma, which is a specialization of [24, Theorem A.14].

**Theorem 10.5.** *Let  $n \geq 4$ . Let  $(W; \partial_0 W, \partial_1 W)$  be a manifold triad of dimension  $(2n+1)$  and let  $(M; \partial_0 M, \partial_1 M)$  and  $(N; \partial_0 N, \partial_1 N)$  be manifold triads of dimension  $n+1$ . Let*

$$f : (M; \partial_0 M, \partial_1 M) \longrightarrow (W; \partial_0 W, \partial_1 W) \quad \text{and} \quad g : (N; \partial_0 N, \partial_1 N) \longrightarrow (W; \partial_0 W, \partial_1 W)$$

*be embeddings such that  $f(\partial_0 M) \cap g(\partial_0 N) = \emptyset$ . Suppose that:*

- *$(W, \partial_1 W)$  is 2-connected;*
- *$(M, \partial_1 M)$  and  $(N, \partial_1 N)$  are both 1-connected.*

*Then there exists an isotopy of embeddings*

$$f_t : (M; \partial_0 M, \partial_1 M) \longrightarrow (W; \partial_0 W, \partial_1 W), \quad t \in [0, 1],$$

*with  $f_0 = f$  and  $f_t|_{\partial_0 M} = f$  for all  $t \in [0, 1]$ , such that  $f_1(M) \cap g(N) = \emptyset$ .*

The following proposition is an application of the above lemma and should be compared to [1, Proposition 6.9]. For what follows let  $n \geq 4$  and let  $M_0$  and  $M_1$  be  $(n-1)$ -connected,  $2n$ -dimensional, closed manifolds. Let  $W$  be a cobordism between  $M_0$  and  $M_1$ . Suppose further that  $W$  is  $(n-1)$ -connected as well.

**Proposition 10.6.** *Let*

$$f, g_1, \dots, g_k : (S^n \times [0, 1], S^n \times \{0, 1\}) \longrightarrow (W, M_0 \sqcup M_1)$$

be a collection of embeddings and let  $x, y_1, \dots, y_k \in H_n(M_0)$  denote the classes represented by

$$f|_{S^n \times \{0\}}, g_1|_{S^n \times \{0\}}, \dots, g_k|_{S^n \times \{0\}}$$

respectively. Let  $K \subset W$  be a submanifold of codimension  $\geq 3$ . Suppose that the following conditions are met:

- (a)  $\lambda(x, y_i) = 0$  for all  $i = 1, \dots, k$ ;
- (b) the embeddings  $g_1, \dots, g_k$  are pairwise transverse.
- (c) the images of  $f$  and  $g_1, \dots, g_k$  are contained in the complement,  $W \setminus K$ .

Then there exists an isotopy  $f_t : (S^n \times [0, 1], S^n \times \{0, 1\}) \rightarrow (W, M_0 \sqcup M_1)$  with  $t \in [0, 1]$ , that satisfies:

- $f_0 = f$ ,
- $f_t(S^n \times [0, 1]) \subset W \setminus K$  for all  $t \in [0, 1]$ ,
- $f_1(S^n \times [0, 1]) \cap g_i(S^n \times [0, 1]) = \emptyset$  for all  $i = 1, \dots, k$ .

Suppose further that  $f$  is such that  $f(S^n \times \{0\}) \cap g_i(S^n \times \{0\}) = \emptyset$  for all  $i = 1, \dots, k$ . Then the isotopy  $f_t$  can be chosen so that  $f_t|_{S^n \times \{0\}} = f|_{S^n \times \{0\}}$  for all  $t \in [0, 1]$ .

*Proof.* By condition (a) we may apply the *Whitney trick* inductively to obtain an isotopy of  $f|_{S^n \times \{0\}}$ , that pushes  $f(S^n \times \{0\}) \subset M_0$  off of the submanifolds  $g_1(S^n \times \{0\}), \dots, g_k(S^n \times \{0\}) \subset M_0$ , while staying in the complement,  $M_0 \setminus (M_0 \cap K)$ . Thus we reduce to the case where

$$f(S^n \times \{0\}) \cap g_i(S^n \times \{0\}) = \emptyset \quad \text{for all } i = 1, \dots, k.$$

(We remark that in order to inductively apply the Whitney trick as we did above, it is necessary that the submanifolds  $g_1(S^n \times \{0\}), \dots, g_k(S^n \times \{0\}) \subset M_0$  be pairwise transverse, see [1, Proposition 6.9].)

Let  $W'$ ,  $M'_0$ , and  $M'_1$  denote the complements  $W \setminus K$ ,  $M_0 \setminus (M_0 \cap K)$ , and  $M_1 \setminus (M_1 \cap K)$ . Since the codimension of  $K$  is greater than or equal to 3, it follows that the pair  $(W', M'_0)$  is 2-connected. To prove the corollary, it will suffice to construct an isotopy

$$f_t : (S^n \times [0, 1], S^n \times \{0, 1\}) \rightarrow (W', M'_0 \sqcup M'_1), \quad t \in [0, 1],$$

with

$$f_0 = f \quad \text{and} \quad f_t|_{S^n \times \{0\}} = f|_{S^n \times \{0\}} \quad \text{for all } t \in [0, 1],$$

such that

$$f(S^n \times [0, 1]) \cap g_i(S^n \times [0, 1]) = \emptyset$$

for all  $i = 1, \dots, k$ . Such an isotopy exists by inductive application of Lemma 10.5, using the fact that the embeddings  $g_1, \dots, g_n$  are pairwise pairwise transverse. We refer to the reader to [1, Proposition 6.9] for precise details on how this induction is carried out using the fact that the embeddings  $g_1, \dots, g_n$  are pairwise pairwise transverse. This concludes the proof of the corollary.  $\square$

The next proposition establishes condition (iii)\*. This should be compared to [1, Lemma 6.10].

**Proposition 10.7.** *For  $p \in \mathbb{Z}_{\geq 0}$ , let  $x = (a, \varepsilon, (W, \ell_W), V) \in \tilde{\mathbf{D}}_{p, -1}^{\mathcal{L}, n}$ .*

- *Let  $\{v_1, \dots, v_k\} \in \tilde{\mathbf{Y}}_0(x)$  be a non-empty collection of elements in general position (i.e pairwise in  $\mathcal{T}$ ).*
- *Let  $\{w_1, \dots, w_s\} \in \tilde{\mathbf{Y}}_0(x)$  be an arbitrary collection of elements with  $(v_i, w_j) \in \tilde{\mathbf{Y}}_1(x)$  for all  $i, j$ .*

*Then there exists  $u \in \tilde{\mathbf{Y}}_0(x)$  such that  $(u, v_j) \in \tilde{\mathbf{Y}}_1(x)$  and  $(u, w_j) \in \tilde{\mathbf{Y}}_1(x)$  for all  $i, j$ .*

*Proof.* For each  $j = 1, \dots, k$ , let  $(\Lambda_j^v, \delta_j^v, e_j^v, \ell_j^v)$  denote the element  $v_j$ , and for  $r = 1, \dots, s$  let  $(\Lambda_r^w, \delta_r^w, e_r^w, \ell_r^w)$  denote the element  $w_r$ . We temporarily set  $u = v_1$  and write  $u = (\Lambda, \delta, e, \ell)$ . Since  $(w_r, v_j) \in \tilde{\mathbf{Y}}_1(x)$  for all  $r, j$ , it follows that

$$e_j^v(\Lambda_j^v \times C) \cap e_r^w(\Lambda_r^w \times C) = \emptyset \quad \text{for all } j, r,$$

where  $C$  is the core from Definition 10.3. Let  $K \subset W$  denote the union  $\cup_{r=1}^s e_r^w(C) \subset W$  and let  $W'$  denote  $W \setminus K$ . We have  $e(\Lambda \times C) \subset W'$  and  $e_j^v(\Lambda_j^v \times C) \subset W'$  for all  $j = 1, \dots, k$ .

By Remark 9.4 the homology classes in  $H_n(W|_{a_0})$  determined by the submanifolds

$$e_j^v(\Lambda_j^v \times C) \cap W|_{a_0} \subset W|_{a_0} \quad j = 1, \dots, k,$$

are all contained in the subspace  $V_0|_{a_0} \subset H_n(W|_{a_0})$ , which is Lagrangian by definition. Similarly, for each  $\lambda \in \delta^{-1}(i)$ , the homology class determined by the submanifold

$$e(\lambda \times C) \cap W|_{a_0} \subset W|_{a_0},$$

is contained in  $V_i|_{a_0} \subset H_n(W|_{a_0})$  as well. Since the intersection form vanishes on  $V_0$ , and the set  $\{v_1, \dots, v_k\}$  is in general position, it follows from Corollary 10.6 that for each  $\lambda \in \Lambda$ , there exists an isotopy of  $e(\lambda \times C)$  that pushes  $e(\lambda \times C) \cap W|_{[a_0, a_1]}$  off of  $e_j^v(\Lambda_j^v \times C) \cap W|_{[a_0, a_1]}$  for all  $j = 1, \dots, k$ , and keeps  $e(\lambda \times C)$  inside of  $W' = W \setminus K \subset W$ . With  $e(\lambda \times C)$  made disjoint from  $e_j^v(\Lambda_j^v \times C) \cap W|_{[a_0, a_1]}$  for all  $j$ , we may re-apply this same procedure to construct an isotopy of  $e(\lambda \times C)$  that pushes  $e(\lambda \times C) \cap W|_{[a_1, a_2]}$  off of  $e_j^v(\Lambda_j^v \times C) \cap W|_{[a_1, a_2]}$  for all  $j$  while keeping  $e(\lambda \times C)$  contained in  $W'$ . Furthermore, by the second statement in Corollary 10.6 we may assume that this isotopy is fixed on  $e(\lambda \times C) \cap W|_{[a_0, a_1]} \subset W|_{[a_0, a_1]}$  so as not to interfere with our first step of making  $e(\lambda \times C) \cap W|_{[a_0, a_1]}$  disjoint from  $e_j^v(\Lambda_j^v \times C) \cap W|_{[a_0, a_1]}$  for all  $j$ . By combining these two steps we have made  $e(\lambda \times C) \cap W|_{[a_0, a_2]}$  disjoint from  $e_j^v(\Lambda_j^v \times C) \cap W|_{[a_0, a_2]}$  for all  $j$ . Continuing this process inductively we arrange for  $e(\lambda \times C) \cap e_j^v(\Lambda_j^v \times C) = \emptyset$  for all  $j$ . We then proceed inductively to repeat this same construction for all of the other elements in  $\Lambda$ . The resulting element  $u = (e, \Lambda, \delta, \ell)$  satisfies

$$e(\Lambda \times C) \cap e_j^v(\Lambda_j^v \times C) = \emptyset \quad \text{and} \quad e(\Lambda \times C) \cap e_r^w(\Lambda_r^w \times C) = \emptyset$$

for all  $j, r$ . □



**10.4. Condition (ii).** To prove Condition (ii), it will suffice to show that  $\widetilde{\mathbf{Y}}_0(x)$  is non-empty for any  $p$ -simplex  $x \in \mathbf{D}_{p,-1}^{\mathcal{L},n}$ . This will require us to develop some preliminary results. The technical tools involved include the embedding theorems of Haefliger and Hudson from [13] and [14]. Both of these embedding theorems require the manifolds involved to be above a certain dimension, and this, along with the higher-dimensional half-Whitney trick employed in the previous section, yields a restriction on  $n$ . There is only one case not excluded by our need for quadratic refinements that has to be excluded because of our use of these theorems: 5-manifolds (and of course 3-manifolds for which a lot of the techniques employed before also fail).

For what follows let  $n \geq 4$  and let  $M_0$  and  $M_1$  be  $(n-1)$ -connected,  $2n$ -dimensional, closed manifolds. Let  $W$  be a cobordism between  $M_0$  and  $M_1$ . Suppose further that  $W$  is  $(n-1)$ -connected as well.

**Proposition 10.8.** *Let  $W$ ,  $M_0$ , and  $M_1$  be as above. Then any map*

$$f : (S^n \times [0, 1], S^n \times \{0, 1\}) \longrightarrow (W, M_0 \sqcup M_1)$$

*is homotopic to an embedding.*

*Proof.* Since  $M_0$  simply connected and  $\dim(M_0) = 2n \geq 8$ , Haefliger's embedding theorem from [13] implies that the restriction map  $f|_{S^n \times \{0\}} : S^n \times \{0\} \longrightarrow M_0$  is homotopic to an embedding. Thus, we may assume that  $f|_{S^n \times \{0\}}$  is an embedding. Since  $\dim(W) \geq 9$  and the pair  $(W, M_1)$  is  $(n-1)$ -connected, we may then apply [14, Theorem 1] to obtain a homotopy

$$f_t : (S^n \times [0, 1], S^n \times \{0, 1\}) \longrightarrow (W, M_0 \sqcup M_1), \quad t \in [0, 1],$$

with  $f_0 = f$  and  $f_t|_{S^n \times \{0\}} = f|_{S^n \times \{0\}}$  for all  $t \in [0, 1]$ , such that  $f_1$  is an embedding.  $\square$

**Proposition 10.9.** *Let  $W$ ,  $M_0$ , and  $M_1$  be as above. Every relative homology class*

$$x \in H_{n+1}(W, M_0 \sqcup M_1)$$

*is represented by an embedding  $(S^n \times [0, 1], S^n \times \{0, 1\}) \longrightarrow (W, M_0 \sqcup M_1)$ .*

*Proof.* Let  $x \in H_{n+1}(W, M_0 \sqcup M_1)$  be as in the statement of the proposition. Consider the boundary map

$$(10.3) \quad \partial : H_{n+1}(W, M_0 \sqcup M_1) \longrightarrow H_n(M_0 \sqcup M_1),$$

and let  $y$  denote the class  $\partial(x)$ . By the Hurewicz theorem (applied to  $\pi_n(M_0)$  and  $\pi_n(M_1)$ ), the class  $y$  is represented by a map

$$\phi : S^n \times \{0, 1\} \longrightarrow M_0 \sqcup M_1$$

sending  $S^n \times \{i\}$  into  $M_i$  for  $i = 0, 1$ . Let  $\iota_i : M_i \hookrightarrow W$  denote the inclusion. By exactness, the class  $y$  maps to zero under  $H_n(M_0 \sqcup M_1) \longrightarrow H_n(W)$ . It follows that the maps

$$\iota_0 \circ \phi|_{S^n \times \{0\}}, \quad -\iota_1 \circ \phi|_{S^n \times \{1\}} : S^n \longrightarrow W$$

are homotopic, where  $-\iota_1 \circ \phi|_{S^n \times \{1\}}$  denotes the pre-composition of  $\iota_1 \circ \phi|_{S^n \times \{1\}}$  with some reflection (which reverses orientation). It then follows that there exists a map

$$\Phi : (S^n \times [0, 1], S^n \times \{0, 1\}) \longrightarrow (W, M_0 \sqcup M_1)$$

such that

$$\Phi|_{S^n \times \{0\}} = \iota_0 \circ \phi|_{S^n \times \{0\}} \quad \text{and} \quad \Phi|_{S^n \times \{1\}} = -\iota_1 \circ \phi|_{S^n \times \{1\}}.$$

By Proposition 10.8 we may deform  $\Phi$ , to a new map  $\Phi'$  such that  $\Phi'$  is an embedding. Let

$$w \in H_{n+1}(W, M_0 \sqcup M_1)$$

denote the class represented by this embedding  $\Phi'$ . It follows that

$$\partial w = y = \partial x.$$

Let  $v$  denote the difference  $w - x \in H_{n+1}(W, M_0 \sqcup M_1)$ . The class  $v$  is in the kernel of  $\partial$  and thus is in the image of  $H_{n+1}(W) \longrightarrow H_{n+1}(W, M_0 \sqcup M_1)$ , and so  $v$  is represented by a map

$$h : S^{n+1} \longrightarrow W.$$

Since  $W$  is  $(n-1)$ -connected, by Haefliger's embedding theorem [13] we may assume that the map  $h$  is an embedding as well. Let

$$\Psi : (S^n \times [0, 1], S^n \times \{0, 1\}) \longrightarrow (W, M_0 \sqcup M_1)$$

be the embedding constructed by forming the connected sum of the image of  $\Phi'$  with the image of  $-h$ , along an embedded arc. The map  $\Psi$  represents the class  $w - v = w - (w - x) = x$ , and thus equals  $x$ . This concludes the proof of the proposition.  $\square$

Recall the manifold  $K^n \subset \mathbb{R}^{n+1} \times D^{n+1}$  from (8.1), used in Definition 9.3. We will need one more preliminary result regarding  $\theta$ -structures on  $K$ . Recall that the restricted manifold  $K^n|_{(-6, -2)}$  agrees with the product  $(-6, -2) \times \mathbb{R}^n \times S^n$ . The following lemma is the only place in the paper where the condition that  $\theta : B \longrightarrow BO(2n+1)$  is weakly once stable is used. The proof follows the exact argument given in the proof of [9, Proposition 6.22 (page 350)] and so we omit the proof.

**Lemma 10.10.** *Suppose that  $\theta : B \longrightarrow BO(2n+1)$  is weakly once stable. Let  $\ell$  be a  $\theta$ -structure on  $K^n|_{(-6, -2)}$ . Suppose that the restriction of  $\ell$  to  $K^n|_{-3} = \{-3\} \times \mathbb{R}^n \times S^n$  extends to a  $\theta$ -structure on  $\{-3\} \times \mathbb{R}^n \times D^{n+1}$ . Then there exists a  $\theta$ -structure  $\ell'$  on  $K^n$  such that  $\ell'|_{(-6, -2)} = \ell|_{(-6, -2)}$ .*

Condition (ii) follows from the proposition below. The proof requires the use of Lemma 10.10 and so there it requires the condition that  $\theta : B \longrightarrow BO(2n+1)$  is weakly once stable.

**Proposition 10.11.** *For  $p \in \mathbb{Z}_{\geq 0}$ , let  $x = (a, \varepsilon, (W, \ell_W), V) \in \mathbf{D}_{p, -1}^{\mathcal{L}^n}$ . The set  $\tilde{\mathbf{Y}}_0(x)$  is non-empty.*

*Proof.* As usual we write  $V = (V_0, \dots, V_p)$ . For each  $j = 0, \dots, p$ , we write  $M_j = W|_{a_j}$ . Let  $i \in \{0, \dots, p\}$ . Consider the subspace  $V_i|_{[a_i, a_p]} \leq H_{n+1}(W|_{[a_i, a_p]}, M_i \sqcup M_p)$ . It follows from the definitions that  $V_i|_{a_i}$  is equal to the image of  $V_i|_{[a_i, a_p]}$  under the homomorphism

$$H_{n+1}(W|_{[a_i, a_p]}, M_i \sqcup M_p) \xrightarrow{\partial} H_n(M_i \sqcup M_p) \cong H_n(M_i) \oplus H_n(M_p) \xrightarrow{\text{pr}_*} H_n(M_i),$$

and thus every element of the Lagrangian subspace  $V_i|_{a_i}$  is equal to the image of some element in  $H_{n+1}(W|_{[a_i, a_p]}, M_i \sqcup M_p)$ , under the above map. By combining Proposition 10.9 with Lemma 4.5, there exists a finite set  $\Lambda_i$  and an embedding

$$(10.4) \quad \varphi_i : \Lambda_i \times [a_i, a_p] \times S^n \longrightarrow W|_{[a_i, a_p]}$$

with the following properties:

- (a) The homology classes represented by the embeddings

$$\varphi_i|_{\lambda \times [a_i, a_p] \times S^n} : \lambda \times [a_i, a_p] \times S^n \longrightarrow W|_{[a_i, a_p]}, \quad \lambda \in \Lambda_i,$$

are all contained in the subspace  $V_i|_{[a_i, a_p]} \leq H_{n+1}(W|_{[a_i, a_p]}, M_i \sqcup M_p)$ .

- (b) The collection of embeddings

$$\varphi_i|_{\lambda \times \{a_i\} \times S^n} : \Lambda_i \times \{a_i\} \times S^n \longrightarrow M_i, \quad \lambda \in \Lambda_i,$$

yields a basis for the subspace  $V_i \leq H_n(M_i)$ .

- (c) The restriction  $\varphi_i|_{\Lambda_i \times \{a_i\} \times S^n} : \Lambda_i \times \{a_i\} \times S^n \longrightarrow M_i$  extends to an embedding

$$\varphi' : \Lambda_i \times \{a_i\} \times \mathbb{R}^n \times S^n \longrightarrow M_i,$$

with the property that the induced bundle map

$$T(\Lambda_i \times \{a_i\} \times \mathbb{R}^n \times S^n) \oplus \epsilon^1 \longrightarrow TM_i \times \epsilon^1 \xrightarrow{\ell_W|_{M_i}} \theta^* \gamma^{2n+1},$$

admits an extension to a  $\theta$ -structure on  $\Lambda_i \times \{a_i\} \times \mathbb{R}^n \times D^{n+1}$ .

The embedding

$$(\Lambda_i \times \{a_i\} \times \mathbb{R}^n \times S^n) \cup (\Lambda_i \times [a_i, a_p] \times S^n) \longrightarrow W|_{[a_i, a_p]}$$

obtained by combining  $\varphi_i$  and  $\varphi'_i$ , extends to an embedding

$$\bar{\varphi}_i : \Lambda_i \times [a_i, a_p] \times \mathbb{R}^n \times S^n \longrightarrow W|_{[a_i, a_p]},$$

and in turn this embedding extends to an embedding

$$(10.5) \quad \hat{\varphi}_i : \Lambda_i \times \mathbb{R} \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^n \times D^{n+1} \longrightarrow \mathbb{R} \times (-1, 1)^{\infty-1}.$$

By carrying out the exact same construction for all  $i = 0, \dots, p$ , we obtain embeddings  $\hat{\varphi}_0, \dots, \hat{\varphi}_p$  as in (10.5). By applying Corollary 10.6, we may arrange for

$$\hat{\varphi}_i(\Lambda_i \times \mathbb{R} \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \{0\} \times S^n) \cap \hat{\varphi}_j(\Lambda_j \times \mathbb{R} \times (a_j - \varepsilon_j, a_p + \varepsilon_p) \times \{0\} \times S^n) = \emptyset$$

for all  $i, j = 0, \dots, p$ . Let  $\Lambda = \sqcup_{i=0}^p \Lambda_i$ . By forming the disjoint union of the embeddings  $\widehat{\varphi}_0, \dots, \widehat{\varphi}_p$  and applying the reparametrization  $(a_i - \varepsilon_i, a_p + \varepsilon_p) \cong (-6, -2)$  from (9.3), we obtain an embedding

$$e : \Lambda \times \mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times D^{n+1} \longrightarrow \mathbb{R} \times (-1, 1)^{\infty-1}$$

that satisfies all of the conditions of Definition 9.3.

The embedding  $e$  determines part of the data of an element of  $\widetilde{\mathbf{Y}}_0(x)$ . The other part of the necessary data is a  $\theta$ -structure  $\ell$  on  $\Lambda \times K|_{(-6, -2)}$  that restricts to the  $\theta$ -structure on

$$\Lambda \times K|_{(-6, -2)} = (-6, -2) \times \mathbb{R}^{n+1} \times S^{n-1},$$

given by the composition

$$(10.6) \quad T(\Lambda \times K|_{(-6, -2)}) \xrightarrow{De} W \xrightarrow{\ell_W} \theta^* \gamma^{2n+1}.$$

This extension exists by Lemma 10.10. In order to apply Lemma 10.10 it is required that the restriction of (10.6) to  $\Lambda \times K|_{-3} = \Lambda \times \{-3\} \times \mathbb{R}^n \times S^n$  admits an extension to a  $\theta$ -structure on  $\Lambda \times \{3\} \times \mathbb{R}^n \times D^{n+1}$ . The existence of this extension is provided by condition (c) of the embedding  $\varphi$  from (10.4). This concludes the proof of the proposition.  $\square$

Propositions 10.7 and 10.11 establish conditions (ii) and (iii)\* for the augmented topological flag complex  $\widetilde{\mathbf{D}}_{p, \bullet}^{\mathcal{L}, n} \longrightarrow \widetilde{\mathbf{D}}_{p, -1}^{\mathcal{L}, n}$  for each  $p \in \mathbb{Z}_{\geq 0}$ , thus it follows that the induced map  $|\widetilde{\mathbf{D}}_{p, \bullet}^{\mathcal{L}, n}| \longrightarrow |\widetilde{\mathbf{D}}_{p, -1}^{\mathcal{L}, n}|$  is a weak homotopy equivalence for each  $p$ . By applying geometric realization to the first coordinate, we obtain the weak homotopy equivalence  $|\widetilde{\mathbf{D}}_{\bullet, \bullet}^{\mathcal{L}, n}| \simeq |\widetilde{\mathbf{D}}_{\bullet, -1}^{\mathcal{L}, n}|$  and thus the weak homotopy equivalence  $|\mathbf{D}_{\bullet, \bullet}^{\mathcal{L}, n}| \simeq |\mathbf{D}_{\bullet, -1}^{\mathcal{L}, n}|$ . This concludes the proof of Theorem 9.6.

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